Introduction to Partial Differentail Equations Peter J. Olver

CH2 Linear and Nonlinear Waves

2.1 Stationary Waves

2.1.1. Solve the partial differential equation $\dfrac{\partial u}{\partial t} = x$ for $u(t,x)$.

2.1.2. Solve the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} = 0$ for $u(t,x)$.

2.1.3. Find the general solution $u(t,x)$ to the following partial differential equations: (a) $u_x = 0$, (b) $u_t = 1$, (c) $u_t = x - t$, (d) $u_t + 3u = 0$, (e) $u_x + t\,u = 0$, (f) $u_{tt} + 4u = 1$.

2.1.4. Suppose $u(t,x)$ is defined for all $(t,x) \in \mathbb{R}^2$ and solves $\partial u/\partial t + 2u = 0$. Prove that $\lim_{t\to\infty} u(t,x) = 0$ for all x.

2.1.5. Write down the general solution to the partial differential equation $\partial u/\partial t = 0$ for a function of three variables $u(t,x,y)$. What assumptions should be made on the domain of definition for your solution formula to be valid?

2.1.6. Solve the partial differential equation $\dfrac{\partial^2 u}{\partial x\,\partial y} = 0$ for $u(x,y)$.

2.1.7. Answer Exercise 2.1.6 when $u(x,y,z)$ depends on the three independent variables x, y, z.

2.1.8. Let $u(t,x)$ solve the initial value problem $\dfrac{\partial u}{\partial t} + u^2 = 0$, $u(0,x) = f(x)$, where $f(x)$ is a bounded C^1 function of $x \in \mathbb{R}$. (a) Show that if $f(x) \geq 0$ for all x, then $u(t,x)$ is defined for all $t > 0$, and $\lim_{t\to\infty} u(t,x) = 0$. (b) On the other hand, if $f(x) < 0$, then the solution $u(t,x)$ is not defined for all $t > 0$, but in fact, $\lim_{t\to\tau^-} u(t,x) = -\infty$ for some $0 < \tau < \infty$. Given x, what is the corresponding value of τ? (c) Given $f(x)$ as in part (b), what is the longest time interval $0 < t < t_\star$ on which $u(t,x)$ is defined for all $x \in \mathbb{R}$?

2.1.9. Justify the claim in the text that if $u(t,x)$ is a solution of $\partial u/\partial t = 0$ that is defined on a domain $D \subset \mathbb{R}^2$ with the property that $D_a = D \cap \{\,(a,x)\,|\,x \in \mathbb{R}\,\}$ is either empty or a connected interval, then $u(t,x) = v(x)$ depends only on $x \in D$.

2.1.10. Prove that the function in (2.3) is continuously differentiable at all points (t,x) in its domain of definition.

2.2 Transport and Traveling Waves

2.2.1. Find the solution to the initial value problem $u_t + u_x = 0,\ \ u(1,x) = x/(1+x^2)$.

2.2.2. Solve the following initial value problems and graph the solutions at times $t = 1, 2$, and 3:
(a) $u_t - 3u_x = 0,\ u(0,x) = e^{-x^2}$; (b) $u_t + 2u_x = 0,\ u(-1,x) = x/(1+x^2)$;
(c) $u_t + u_x + \frac{1}{2}u = 0,\ u(0,x) = \tan^{-1} x$; (d) $u_t - 4u_x + u = 0,\ u(0,x) = 1/(1+x^2)$.

2.2.3. Graph some of the characteristic lines for the following equations, and write down a formula for the general solution:
(a) $u_t - 3u_x = 0$, (b) $u_t + 5u_x = 0$, (c) $u_t + u_x + 3u = 0$, (d) $u_t - 4u_x + u = 0$.

2.2.4. Solve the initial value problem $u_t + 2u_x = 1,\ \ u(0,x) = e^{-x^2}$.
Hint: Use characteristic coordinates.

2.2.5. Answer Exercise 2.2.4 for the initial value problem $u_t + 2u_x = \sin x,\ \ u(0,x) = \sin x$.

2.2.6. Let c be constant. Suppose that $u(t,x)$ solves the initial value problem $u_t + c u_x = 0$, $u(0,x) = f(x)$. Prove that $v(t,x) = u(t - t_0, x)$ solves the initial value problem $v_t + c v_x = 0$, $v(t_0, x) = f(x)$.

2.2.7. Is Exercise 2.2.6 valid when the transport equation is replaced by the damped transport equation (2.14)?

2.2.8. Let $c \neq 0$. (a) Prove that if the initial data satisfies $u(0,x) = v(x) \to 0$ as $x \to \pm\infty$, then, for each fixed x, the solution to the transport equation (2.4) satisfies $u(t,x) \to 0$ as $t \to \infty$. (b) Is the convergence uniform in x?

2.2.9. (a) Prove that if the initial data is bounded, $|\,f(x)\,| \leq M$ for all $x \in \mathbb{R}$, then the solution to the damped transport equation (2.14) with $a > 0$ satisfies $u(t,x) \to 0$ as $t \to \infty$.
(b) Find a solution to (2.14) that is defined for all (t,x) but does not satisfy $u(t,x) \to 0$ as $t \to \infty$.

2.2.10. Let $F(t,x)$ be a C^1 function of $(t,x) \in \mathbb{R}^2$. (a) Write down a formula for the general solution $u(t,x)$ to the inhomogeneous partial differential equation $u_t = F(t,x)$.
(b) Solve the inhomogeneous transport equation $u_t + c\,u_x = F(t,x)$.

2.2.11. (a) Write down a formula for the general solution to the nonlinear partial differential equation $u_t + u_x + u^2 = 0$. (b) Show that if the initial data is positive and bounded, $0 \le u(0,x) = f(x) \le M$, then the solution exists for all $t > 0$, and $u(t,x) \to 0$ as $t \to \infty$. (c) On the other hand, if the initial data is negative at some x, then the solution *blows up* at x in finite time: $\lim_{t \to \tau^-} u(t,x) \to -\infty$ for some $\tau > 0$. (d) Find a formula for the earliest blow-up time $\tau_\star > 0$.

2.2.12. A sensor situated at position $x = 1$ monitors the concentration of a pollutant $u(t,1)$ as a function of t for $t \ge 0$. Assuming that the pollutant is transported with wave speed $c = 3$, at what locations x can you determine the initial concentration $u(0,x)$?

2.2.13. Write down a solution to the transport equation $u_t + 2u_x = 0$ that is defined on a connected domain $D \subset \mathbb{R}^2$ and that is *not* a function of the characteristic variable alone.

2.2.14. Let $c > 0$. Consider the uniform transport equation $u_t + c u_x = 0$ restricted to the quarter-plane $Q = \{x > 0,\ t > 0\}$ and subject to initial conditions $u(0,x) = f(x)$ for $x \ge 0$, along with boundary conditions $u(t,0) = g(t)$ for $t \ge 0$. (a) For which initial and boundary conditions does a classical solution to this initial-boundary value problem exist? Write down a formula for the solution. (b) On which regions are the effects of the initial conditions felt? What about the boundary conditions? Is there any interaction between the two?

2.2.15. Answer Exercise 2.2.14 when $c < 0$.

Nonuniform Transport

2.2.16. (a) Find the general solution to the first-order equation $u_t + \frac{3}{2} u_x = 0$.
(b) Find a solution satisfying the initial condition $u(1,x) = \sin x$. Is your solution unique?

2.2.17. (a) Solve the initial value problem $u_t - x u_x = 0$, $u(0,x) = (x^2+1)^{-1}$.
(b) Graph the solution at times $t = 0, 1, 2, 3$. (c) What is $\lim_{t \to \infty} u(t,x)$?

2.2.18. Suppose the initial data $u(0,x) = f(x)$ of the nonuniform transport equation (2.28) is continuous and satisfies $f(x) \to 0$ as $|x| \to \infty$. What is the limiting solution profile $u(t,x)$ as (a) $t \to \infty$? (b) $t \to -\infty$?

2.2.19. (a) Find and graph the characteristic curves for the equation $u_t + (\sin x) u_x = 0$.
(b) Write down the solution with initial data $u(0,x) = \cos \frac{1}{2} \pi x$. (c) Graph your solution at times $t = 0, 1, 2, 3, 5$, and 10. (d) What is the limiting solution profile as $t \to \infty$?

2.2.20. Consider the linear transport equation $u_t + (1 + x^2) u_x = 0$. (a) Find and sketch the characteristic curves. (b) Write down a formula for the general solution. (c) Find the solution to the initial value problem $u(0,x) = f(x)$ and discuss its behavior as t increases.

2.2.21. Prove that, for $t \gg 0$, the speed of the wave in Example 2.4 is asymptotically proportional to $t^{-2/3}$.

2.2.22. Verify directly that formula (2.21) defines a solution to the differential equation (2.16).

2.2.23. Explain how to derive the solution formula (2.30). Justify that it defines a solution to equation (2.28).

2.2.24. Let $c(x)$ be a bounded C^1 function, so $|\,c(x)\,| \leq c_\star < \infty$ for all x. Let $f(x)$ be any C^1 function. Prove that the solution $u(t,x)$ to the initial value problem $u_t + c(x)\,u_x = 0$, $u(0,x) = f(x)$, is uniquely defined for all $(t,x) \in \mathbb{R}^2$.

2.2.25. Suppose that $c(x) \in \mathrm{C}^1$ is continuously differentiable for all $x \in \mathbb{R}$. (a) Prove that the characteristic curves of the transport equation (2.16) cannot cross each other. (*b*) A point where $c(x_\star) = 0$ is known as a *fixed point* for the characteristic equation $dx/dt = c(x)$. Explain why the characteristic curve passing through a fixed point $(t, x_\star)$ is a horizontal straight line. (*c*) Prove that if $x = g(t)$ is a characteristic curve, then so are all the horizontally translated curves $x = g(t+\delta)$ for any δ. (*d*) *True or false*: Every characteristic curve has the form $x = g(t+\delta)$, for some fixed function $g(t)$. (*e*) Prove that each non-horizontal characteristic curve is the graph $x = g(t)$ of a strictly monotone function. (*f*) Explain why a wave cannot reverse its direction. (*g*) Show that a non-horizontal characteristic curve starts, in the distant past, $t \to -\infty$, at either a fixed point or at $-\infty$ and ends, as $t \to +\infty$, at either the next-larger fixed point or at $+\infty$.

2.2.26. Consider the transport equation $\dfrac{\partial u}{\partial t} + c(t,x)\,\dfrac{\partial u}{\partial x} = 0$ with time-varying wave speed. Define the corresponding characteristic ordinary differential equation to be $\dfrac{dx}{dt} = c(t,x)$, the graphs of whose solutions $x(t)$ are the *characteristic curves*. (a) Prove that any solution $u(t,x)$ to the partial differential equation is constant on each characteristic curve. (*b*) Suppose that the general solution to the characteristic equation is written in the form $\xi(t,x) = k$, where k is an arbitrary constant. Prove that $\xi(t,x)$ defines a *characteristic variable*, meaning that $u(t,x) = f(\xi(t,x))$ is a solution to the time-varying transport equation for any continuously differentiable scalar function $f \in \mathrm{C}^1$.

2.2.27. (a) Apply the method in Exercise 2.2.26 to find the characteristic curves for the equation $u_t + t^2\,u_x = 0$. (*b*) Find the solution to the initial value problem $u(0,x) = e^{-x^2}$, and discuss its dynamic behavior.

2.2.28. Solve Exercise 2.2.27 for the equation $u_t + (x-t)\,u_x = 0$.

2.2.29. Consider the first-order partial differential equation $u_t + (1-2t)\,u_x = 0$. Use Exercise 2.2.26 to: (a) Find and sketch the characteristic curves. (b) Write down the general solution. (c) Solve the initial value problem with $u(0,x) = \dfrac{1}{1+x^2}$. (d) Describe the behavior of your solution $u(t,x)$ from part (c) as $t \to \infty$. What about $t \to -\infty$?

2.2.30. Discuss which of the conclusions of Exercise 2.2.25 are valid for the characteristic curves of the transport equation with time-varying wave speed, as analyzed in Exercise 2.2.26.

2.2.31. Consider the two-dimensional transport equation $\dfrac{\partial u}{\partial t} + c(x,y)\,\dfrac{\partial u}{\partial x} + d(x,y)\,\dfrac{\partial u}{\partial y} = 0$, whose solution $u(t,x,y)$ depends on time t and space variables x, y. (a) Define a characteristic curve, and prove that the solution is constant along it. (b) Apply the method of characteristics to solve the initial value problem $u_t + y\,u_x - x\,u_y$, $u(0,x,y) = e^{-(x-1)^2-(y-1)^2}$. (c) Describe the behavior of your solution.

2.3 Nonlinear Transport and Shocks

2.3.1. Discuss the behavior of the solution to the nonlinear transport equation (2.31) for the following initial data:

(a) $u(0,x) = \begin{cases} 2, & x < -1, \\ 1, & x > -1; \end{cases}$ (b) $u(0,x) = \begin{cases} -2, & x < -1, \\ 1, & x > -1; \end{cases}$ (c) $u(0,x) = \begin{cases} 1, & x < 1, \\ -2, & x > 1. \end{cases}$

2.3.2. Solve the following initial value problems: (a) $u_t + 3\,u\,u_x = 0,\ \ u(0,x) = \begin{cases} 2, & x < 1, \\ 0, & x > 1; \end{cases}$

(b) $u_t - u\,u_x = 0,\ \ u(1,x) = \begin{cases} -1, & x < 0, \\ 3, & x > 0; \end{cases}$ (c) $u_t - 2\,u\,u_x = 0,\ \ u(0,x) = \begin{cases} 1, & x < 1, \\ 0, & x > 1. \end{cases}$

2.3.3. Let $u(0,x) = (x^2+1)^{-1}$. Does the resulting solution to the nonlinear transport equation (2.31) produce a shock wave? If so, find the time of onset of the shock, and sketch a graph of the solution just before and soon after the shock wave. If not, explain what happens to the solution as t increases.

2.3.4. Solve Exercise 2.3.3 when $u(0,x) =$ (a) $-(x^2+1)^{-1}$, (b) $x(x^2+1)^{-1}$.

2.3.5. Consider the initial value problem $u_t - 2\,u\,u_x = 0$, $u(0,x) = e^{-x^2}$. Does the resulting solution produce a shock wave? If so, find the time of onset of the shock and the position at which it first forms. If not, explain what happens to the solution as t increases.

2.3.6. (a) For what values of $\alpha, \beta, \gamma, \delta$ is $u(t,x) = \dfrac{\alpha x + \beta}{\gamma t + \delta}$ a solution to (2.31)?

(b) For what values of $\alpha, \beta, \gamma, \delta, \lambda, \mu$ is $u(t,x) = \dfrac{\lambda t + \alpha x + \beta}{\gamma t + \mu x + \delta}$ a solution to (2.31)?

2.3.7. A *triangular wave* is a shock-wave solution to the initial value problem for (2.31) that has initial data $u(0,x) = \begin{cases} m\,x, & 0 \le x \le \ell, \\ 0, & \text{otherwise.} \end{cases}$ Assuming $m > 0$, write down a formula for the triangular-wave solution at times $t > 0$. Discuss what happens to the triangular wave as time progresses.

2.3.8. Solve Exercise 2.3.7 when $m < 0$.

2.3.9. Solve (2.31) for $t > 0$ subject to the following initial conditions, and graph your solution at some representative times. In what sense does your solution conserve mass?

(a) $u(0,x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$ (b) $u(0,x) = \begin{cases} x, & -1 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$

(c) $u(0,x) = \begin{cases} -x, & -1 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$ (d) $u(0,x) = \begin{cases} 1 - |\,x\,|, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

2.3.10. An N*–wave* is a solution to the nonlinear transport equation (2.31) that has initial conditions $u(0,x) = \begin{cases} m\,x, & -\ell \le x \le \ell, \\ 0, & \text{otherwise,} \end{cases}$ where $m > 0$. (*a*) Write down a formula for the N–wave solution at times $t > 0$. (*b*) What about when $m < 0$?

2.3.11. Suppose $u(t,x)$ and $\widetilde{u}(t,x)$ are two solutions to the nonlinear transport equation (2.31) such that, for some $t_\star > 0$, they agree: $u(t_\star,x) = \widetilde{u}(t_\star,x)$ for all x. Do the solutions necessarily have the same initial conditions: $u(0,x) = \widetilde{u}(0,x)$? Use your answer to discuss the uniqueness of solutions to the nonlinear transport equation.

2.3.12. Suppose that $x_1 < x_2$ are such that the characteristic lines of (2.31) through $(0,x_1)$ and $(0,x_2)$ cross at a shock at $(t,\sigma(t))$ and, moreover, the left- and right-hand shock values (2.52) are $f(x_1) = u^-(t)$, $f(x_1) = u^+(t)$. Explain why the signed area of the region between the graph of $f(x)$ and the secant line connecting $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is zero.

2.3.13. Consider the initial value problem $u^\varepsilon(0,x) = 2 + \tan^{-1}(x/\varepsilon)$ for the nonlinear transport equation (2.31). (*a*) Show that, as $\varepsilon \to 0^+$, the initial condition converges to a step function (2.51). What are the values of a, b? (*b*) Show that, moreover, the resulting solution $u^\varepsilon(0,x)$ to the nonlinear transport equation converges to the corresponding rarefaction wave (2.54) resulting from the limiting initial condition.

2.3.14. (*a*) Under what conditions can equation (2.35) be solved for a single-valued function $u(t,x)$? *Hint*: Use the Implicit Function Theorem. (*b*) Use implicit differentiation to prove that the resulting function $u(t,x)$ is a solution to the nonlinear transport equation.

2.3.15. For what values of $\alpha, \beta, \gamma, \delta, k$ is $u(t,x) = \left(\dfrac{\alpha x + \beta}{\gamma t + \delta}\right)^k$ a solution to the transport equation $u_t + u^2\, u_x = 0$?

2.3.16. (a) Solve the initial value problem $u_t + u^2 u_x = 0$, $u(0,x) = f(x)$, by the method of characteristics. (b) Discuss the behavior of solutions and compare/contrast with (2.31).

2.3.17. (a) Determine the Rankine–Hugoniot condition, based on conservation of mass, for the speed of a shock for the equation $u_t + u^2 u_x = 0$. (b) Solve the initial value problem $u(0,x) = \begin{cases} a, & x < 0, \\ b, & x > 0, \end{cases}$ when (i) $|a| > |b|$, (ii) $|a| < |b|$. *Hint*: Use Exercise 2.3.15 to determine the shape of a rarefaction wave.

2.3.18. Solve Exercise 2.3.17 when the wave speed $c(u) =$ (i) $1 - 2u$, (ii) u^3, (iii) $\sin u$.

2.3.19. Justify the shock-speed formula (2.58).

2.3.20. Consider the general quasilinear first-order partial differential equation

$$\frac{\partial u}{\partial t} + c(t,x,u)\frac{\partial u}{\partial x} = h(t,x,u).$$

Let us define a *lifted characteristic curve* to be a solution $(t, x(t), u(t))$ to the system of ordinary differential equations $\dfrac{dx}{dt} = c(t,x,u)$, $\dfrac{du}{dt} = h(t,x,u)$. The corresponding *characteristic curve* $\big(t, x(t)\big)$ is obtained by projecting to the (t,x)–plane. Prove that if $u(t,x)$ is a solution to the partial differential equation, and $u(t_0,x_0) = u_0$, then the lifted characteristic curve passing through (t_0,x_0,u_0) lies on the graph of $u(t,x)$. Conclude that the graph of the solution to the initial value problem $u(t_0,x) = f(x)$ is the union of all lifted characteristic curves passing through the initial data points $\big(t_0, x_0, f(x_0)\big)$.

2.3.21. Let $a > 0$. (a) Apply the method of Exercise 2.3.20 to solve the initial value problem for the *damped transport equation*: $u_t + u\,u_x + a\,u = 0$, $u(0,x) = f(x)$.
(b) Does the damping eliminate shocks?

2.3.22. Apply the method of Exercise 2.3.20 to solve the initial value problem

$$u_t + t\,u_x = u^2, \qquad u(0,x) = \frac{1}{1+x^2}.$$

2.4 The Wave Equation：d’Alembert’s Formula

2.4.1. Solve the initial value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = e^{-x^2}$, $u_t(0,x) = \sin x$.

2.4.2. (a) Solve the wave equation $u_{tt} = u_{xx}$ when the initial displacement is the box function $u(0,x) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$ while the initial velocity is 0.
(b) Sketch the resulting solution at several representative times.

2.4.3. Answer Exercise 2.4.2 when the initial velocity is the box function, while the initial displacement is zero.

2.4.4. Write the following solutions to the wave equation $u_{tt} = u_{xx}$ in d'Alembert form (2.82). *Hint*: What is the appropriate initial data?
(a) $\cos x \cos t$, (b) $\cos 2x \sin 2t$, (c) e^{x+t}, (d) $t^2 + x^2$, (e) $t^3 + 3tx^2$.

2.4.5. (a) Solve the *dam break problem*, that is, the wave equation when the initial displacement is a step function $\sigma(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$ and there is no initial velocity. (b) Analyze the case in which there is no initial displacement, while the initial velocity is a step function. (c) Are your solutions classical solutions? Explain your answer. (d) Prove that the step function is the limit, as $n \to \infty$, of the functions $f_n(x) = \dfrac{1}{\pi} \tan^{-1} nx + \dfrac{1}{2}$. (e) Show that, in both cases, the step function solution can be realized as the limit, as $n \to \infty$, of solutions to the initial value problems with the functions $f_n(x)$ as initial displacement or velocity.

2.4.6. Suppose $u(t,x)$ solves the initial value problem $u(0,x) = f(x)$, $u_t(0,x) = g(x)$, for the wave equation (2.66). Prove that the solution to the initial value problem $u(t_0,x) = f(x)$, $u_t(t_0,x) = g(x)$, is $u(t - t_0, x)$.

2.4.7. Find all resonant frequencies for the wave equation with wave speed c when subject to the external forcing function $F(t,x) = \sin \omega t \, \sin kx$ for fixed $\omega, k > 0$.

2.4.8. Consider the initial value problem $u_{tt} = 4u_{xx} + F(t,x)$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$. Determine (a) the domain of influence of the point $(0,2)$; (b) the domain of dependence of the point $(3,-1)$; (c) the domain of influence of the point $(3,-1)$.

2.4.9. (a) A solution to the wave equation $u_{tt} = 2u_{xx}$ is generated by a displacement concentrated at position $x_0 = 1$ and time $t_0 = 0$, but no initial velocity. At what time will an observer at position $x_1 = 5$ feel the effect of this displacement? Will the observer continue to feel an effect in the future? (b) Answer part (a) when there is an initial velocity concentrated at position $x_0 = 1$ and time $t_0 = 0$, but no initial displacement.

2.4.10. Suppose $u(t,x)$ solves the initial value problem $u_{tt} = 4u_{xx} + \sin \omega t \, \cos x$, $u(0,x) = 0$, $u_t(0,x) = 0$. Is $h(t) = u(t,0)$ a periodic function?

2.4.11. (a) Write down an explicit formula for the solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \qquad u(0,x) = \sin x, \qquad \frac{\partial u}{\partial t}(0,x) = \cos x, \qquad -\infty < x < \infty, \quad t \geq 0.$$

(b) *True or false*: The solution is a periodic function of t.
(c) Now solve the forced initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \cos 2t, \quad u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = \cos x, \qquad -\infty < x < \infty, \quad t \geq 0.$$

(d) *True or false*: The forced equation exhibits resonance. Explain.
(e) Does the answer to part (d) change if the forcing function is $\sin 2t$?

2.4.12. Given a classical solution $u(t,x)$ of the wave equation, let $E = \frac{1}{2}(u_t^2 + c^2 u_x^2)$ be the associated *energy density* and $P = u_t u_x$ the *momentum density*.
(a) Show that both E and P are conserved densities for the wave equation.
(b) Show that $E(t,x)$ and $P(t,x)$ both satisfy the wave equation.

2.4.13. Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$. The *total energy*

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right] dx \tag{2.98}$$

represents the sum of kinetic and potential energies of the displacement $u(t,x)$ at time t. Suppose that $\nabla u \to \mathbf{0}$ sufficiently rapidly as $x \to \pm\infty$; more precisely, one can find $\alpha > \frac{1}{2}$ and $C(t) > 0$ such that $|\,u_t(t,x)\,|,\ |\,u_x(t,x)\,| \leq C(t)/|\,x\,|^\alpha$ for each fixed t and all sufficiently large $|\,x\,| \gg 0$. For such solutions, establish the *Law of Conservation of Energy* by showing that $E(t)$ is finite and constant. *Hint*: You do not need the formula for the solution.

2.4.14. (a) Use Exercise 2.4.13 to prove that the only classical solution to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = 0$, $u_t(0,x) = 0$, satisfying the indicated decay assumptions is the trivial solution $u(t,x) \equiv 0$. (b) Establish the following *Uniqueness Theorem* for the wave equation: there is at most one such solution to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$.

2.4.15. The *telegrapher's equation* $u_{tt} + a\,u_t = c^2 u_{xx}$, with $a > 0$, models the vibration of a string under frictional damping. (a) Show that, under the decay assumptions of Exercise 2.4.13, the wave energy (2.98) of a classical solution is a nonincreasing function of t. (b) Prove uniqueness of such solutions to the initial value problem for the telegrapher's equation.

2.4.16. What happens to the proof of Theorem 2.14 if $c = 0$?

2.4.17. (a) Explain why the d'Alembert factorization method doesn't work when the wave speed $c(x)$ depends on the spatial variable x.
(b) Does it work when $c(t)$ depends only on the time t?

2.4.18. The *Poisson–Darboux equation* is $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{2}{x}\dfrac{\partial u}{\partial x} = 0$. Solve the initial value problem $u(0,x) = 0$, $u_t(0,x) = g(x)$, where $g(x) = g(-x)$ is an even function. *Hint*: Set $w = x\,u$.

2.4.19. (a) Solve the initial value problem $u_{tt} - 2u_{tx} - 3u_{xx} = 0$, $u(0,x) = x^2$, $u_t(0,x) = e^x$. *Hint*: Factor the associated linear differential operator. (*b*) Determine the domain of influence of a point $(0,x)$. (*c*) Determine the domain of dependence of a point (t,x) with $t > 0$.

2.4.20. (a) Use polar coordinates to prove that, for any $a > 0$,

$$\iint_{\mathbb{R}^2} e^{-a(x^2+y^2)}\, dx\, dy = \frac{\pi}{a}. \tag{2.99}$$

(*b*) Explain why

$$\int_{-\infty}^{\infty} e^{-a x^2}\, dx = \sqrt{\frac{\pi}{a}}. \tag{2.100}$$

2.4.21. Use Exercise 2.4.20 to prove the error function formulae (2.88).