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Ch12 PDE in Space

12.1 The Three-dimensional Laplace and Poisson Equations

12.1.1. Find bases for the following: (*a*) the space of harmonic polynomials $u(x,y,z)$ of degree ≤ 2; (*b*) the space of homogeneous cubic harmonic polynomials $u(x,y,z)$.

12.1.2. *True or false*: (*a*) Every harmonic polynomial is homogeneous.
(*b*) Every homogeneous polynomial is harmonic.

12.1.3. Solve the Poisson boundary value problem $-\Delta u = 1$ on the unit ball $x^2+y^2+z^2<1$ with homogeneous Dirichlet boundary conditions. *Hint*: Look for a polynomial solution.

12.1.4. Prove that if $u(x,y,z)$ solves the Laplace equation, then so does the translated function $U(x,y,z) = u(x-a, y-b, z-c)$ for constants a,b,c.

12.1.5. (*a*) Prove that if $u(x,y,z)$ solves Laplace's equation, so does the rescaled function $U(x,y,z) = u(\lambda x, \lambda y, \lambda z)$ for any constant λ. (*b*) More generally, show that $U(x,y,z) = \mu\, u(\lambda x, \lambda y, \lambda z) + c$ solves Laplace's equation for any constants λ, μ, c.

12.1.6. Let A be a constant nonsingular 3×3 matrix, $u(\mathbf{x})$ a C^1 scalar field, and $\mathbf{v}(\mathbf{x})$ a C^1 vector field. Set $U(\mathbf{x}) = u(A\,\mathbf{x})$ and $\mathbf{V}(\mathbf{x}) = \mathbf{v}(A\,\mathbf{x})$. Prove that
(*a*) $\nabla U(\mathbf{x}) = A^T \nabla u(A\,\mathbf{x})$, (*b*) $\nabla\cdot\mathbf{V}(\mathbf{x}) = w(A\,\mathbf{x})$, where $w(\mathbf{x}) = \nabla\cdot(A\mathbf{v})(\mathbf{x})$.

12.1.7. Prove that every rotation and reflection is a symmetry of the Laplace equation. In other words, if Q is any 3×3 orthogonal matrix, so $Q^TQ = \mathrm{I}$, and $u(\mathbf{x})$ is a harmonic function, then so is $U(\mathbf{x}) = u(Q\,\mathbf{x})$. *Hint*: Use Exercise 12.1.6.

12.1.8. *The Weak Maximum Principle*: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $u(x,y,z)$ solve the Poisson equation $-\Delta u = f(x,y,z)$, where $f(x,y,z) < 0$ for all $(x,y,z)\in\Omega$.
(*a*) Prove that the maximum value of u occurs on the boundary $\partial\Omega$.
Hint: Explain why u cannot have a local maximum at any interior point in Ω.
(*b*) Generalize your result to the case $f(x,y,z)\leq 0$.
Hint: Look at $v_\varepsilon(x,y,z) = u(x,y,z) + \varepsilon\,(x^2+y^2+z^2)$ and let $\varepsilon \to 0^+$.

12.1.9. Find the equilibrium equations corresponding to minimizing $|||\,\nabla u\,|||^2$ subject to homogeneous Dirichlet boundary conditions, where the indicated norm is based on the weighted inner product

$$\langle\!\langle\, \mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle = \iiint_\Omega \mathbf{v}(x,y,z)\cdot\mathbf{w}(x,y,z)\,\sigma(x,y,z)\,dx\,dy\,dz,$$

with $\sigma(x,y,z) > 0$ a positive scalar function.

12.1.10. Prove the following vector calculus identities:
(a) $\nabla \cdot (u\,\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\,\nabla \cdot \mathbf{v}$, (b) $\nabla \times (u\,\mathbf{v}) = \nabla u \times \mathbf{v} + u\,\nabla \times \mathbf{v}$,
(c) $\nabla \cdot (\mathbf{v} \times \mathbf{w}) = (\nabla \times \mathbf{v}) \cdot \mathbf{w} - \mathbf{v} \cdot (\nabla \times \mathbf{w})$, (d) $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v}$.
(In the final term, the Laplacian Δ acts component-wise on the vector field $\mathbf{v}$.)

12.1.11. Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$. Prove the following identities: (a) $\iiint_\Omega \Delta u\, dx\, dy\, dz = \iint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\, dS$,
(b) $\iiint_\Omega u\,\Delta u\, dx\, dy\, dz = \iint_{\partial\Omega} u\,\frac{\partial u}{\partial \mathbf{n}}\, dS - \iiint_\Omega \|\, \nabla u \,\|^2\, dx\, dy\, dz$.

12.1.12. Suppose the inhomogeneous Neumann boundary value problem (12.1, 4) has a solution. (a) Prove that $\iint_{\partial\Omega} k\, dS = 0$. (b) Is the solution unique? If not, what is the most general solution? (c) State and prove an analogous result for the inhomogeneous Poisson equation $-\Delta u = f(x, y, z)$. (d) Provide a physical explanation for your answers.

12.1.13. Find a minimization principle that characterizes the solution to the inhomogeneous mixed boundary value problem $-\Delta u = f$ on Ω, with $u = g$ on $D \subsetneq \partial\Omega$, and $\partial u/\partial \mathbf{n} = h$ on $N = \partial\Omega \setminus D$.

12.1.14. (a) Prove that, subject to suitable boundary conditions, the curl $\nabla\times$ defines a self-adjoint operator with respect to the L^2 inner product between vector fields. What kinds of boundary conditions do you need to impose for your integration by parts argument to be valid? *Hint*: Use the identity in Exercise 12.1.10(c). (b) What operator on vector fields is given by the self-adjoint composition $S = (\nabla\times)^* \circ (\nabla\times)$? (c) Choose a set of homogeneous boundary conditions that make S self-adjoint. Is the resulting boundary value problem $S[\mathbf{v}] = \mathbf{f}$ positive definite? If not, what does the Fredholm Alternative say about its solvability?

12.2 Separation of Variables for the Laplace Equation

12.2.1. A solid ball of radius R has its upper hemispherical surface held at temperature T_1 and its lower hemispherical surface held at temperature T_0. Find the resulting equilibrium temperature.

12.2.2. A solid ball has its top hemispherical surface insulated and its bottom hemispherical surface held at a fixed temperature of 10°. Find its equilibrium temperature.

12.2.3. Find the potential inside a spherical capacitor of radius R when the upper hemisphere is at potential α and the lower is at β.

12.2.4. Find the potential $u(x, y, z)$ inside a unit spherical capacitor that has the indicated boundary values on the unit sphere $x^2+y^2+z^2 = 1$: (a) x, (b) x^2+y^2, (c) x^3. *Hint*: The potential is a polynomial.

12.2.5. Each point on the spherical boundary of a solid ball of radius 1 has temperature equal to its zenith angle φ. (a) Find the value of the equilibrium temperature at the center of the ball. (b) Find the Taylor polynomial of degree 3, based at the origin, for the equilibrium temperature distribution.

12.2.6. Solve Exercise 12.2.5 when the boundary temperature equals (a) $\cos\varphi$, (b) $\cos\theta$, (c) θ.

12.2.7. A solid spherical container of radius 3 cm contains a hollow spherical cavity of radius 1 cm in its center. The inner cavity is filled with boiling water at 100°, while the entire container is immersed in an ice water bath at 0°. Assume that the container is in thermal equilibrium. *True or false*: The temperature at a point half-way between the container's inner and outer boundaries is 50°. If true, explain. If false, what is the temperature at such a point?

12.2.8. Find the electrostatic potential between two concentric spherical metal shells of respective radii 1 and 1.2, given that the inner shell is grounded, while the outer shell has potential equal to 1.

12.2.9. Use the chain rule to establish the formula (12.16) for the Laplacian in spherical coordinates.

12.2.10. (a) Prove that $t = \pm 1$ are both regular singular points for the order 0 Legendre differential equation (12.28). (b) Prove that the Legendre eigenvalue problem (12.27–28) is defined by a self-adjoint operator with respect to the L^2 inner product on the cut locus $[-1, 1]$. (c) Discuss the orthogonality of the Legendre polynomials.

12.2.11. Solve Exercise 12.2.10 for the Legendre eigenvalue problem (12.26–27) of order m along with the relevant Ferrers eigenfunctions.

12.2.12. Suppose $m > 0$. (a) Find the Green's function for the boundary value problem

$$(1-t^2)\,\frac{d^2P}{dt^2} - 2t\,\frac{dP}{dt} - \frac{m^2}{1-t^2}\,P = f(t), \qquad |\,P(-1)\,|,\ |\,P(1)\,| < \infty.$$

Hint: The homogeneous differential equation has solutions $\left(\dfrac{1+t}{1-t}\right)^{\frac{m}{2}}$ and $\left(\dfrac{1-t}{1+t}\right)^{\frac{m}{2}}$.
(b) Use part (a) to prove completeness of the Ferrers functions of order $m > 0$ on $[-1, 1]$.
(c) Explain why there is no Green's function in the order $m = 0$ case.
Remark: When $m = 0$, one can use the trick of Example 9.49 to prove completeness. Although the Green's function for the modified operator does not have an explicit elementary formula, one can prove that it has logarithmic singularities at the endpoints, and hence finite double L^2 norm. See [**120**; §43] for details.

12.2.13. What happens when $n < m$ in formula (12.31)?

12.2.14. Prove that the Legendre polynomial (12.29) has the explicit formula

$$P_n(t) = \sum_{0 \le 2m \le n} (-1)^m \frac{(2n-2m)!}{2^n\,(n-m)!\,m!\,(n-2m)!}\,t^{n-2m}. \tag{12.64}$$

12.2.15. Prove the following recurrence relation for the Ferrers functions:

$$P_n^{m+1}(t) = \sqrt{1-t^2}\,\frac{dP_n^m}{dt} + \frac{m\,t}{\sqrt{1-t^2}}\,P_n^m(t). \tag{12.65}$$

12.2.16. In this exercise, we determine the L^2 norms of the Ferrers functions. (*a*) First, prove that $\int_{-1}^{1}(1-t^2)^n\,dt = \dfrac{2^{2n+1}\,(n!)^2}{(2n+1)!}$. *Hint*: Set $t = \cos\theta$ and then integrate by parts repeatedly. (*b*) Prove that $\|\,P_n\,\|^2 = \dfrac{2}{2n+1}$. *Hint*: Integrate by parts repeatedly and then use part (a). (*c*) Prove that $\|\,P_n^{m+1}\,\|^2 = (n-m)(n+m+1)\,\|\,P_n^m\,\|^2$. *Hint*: Use (12.65) and an integration by parts. (*d*) Finally, prove that $\|\,P_n^m\,\|^2 = \dfrac{2}{2n+1}\,\dfrac{(n+m)!}{(n-m)!}$.

12.2.17. (*a*) Prove that $P_n^m(t)$ is an even or odd function according to whether $m+n$ is an even or odd integer. (*b*) Prove that its Fourier form, $p_n^m(\varphi)$, depends only on $\cos n\,\varphi$, $\cos(n-2)\varphi$, $\cos(n-4)\varphi$, $\ldots$ if m is even, and only on $\sin n\,\varphi$, $\sin(n-2)\varphi$, $\sin(n-4)\varphi$, $\ldots$ if m is odd.

12.2.18. Let m be fixed. Are the functions $p_n^m(\varphi)$ for $n = 0, 1, 2, \ldots$ mutually orthogonal with respect to the standard L^2 inner product on $[0, \pi]$? If not, is there an inner product that makes them orthogonal functions?

12.2.19. Prove that the surfaces defined by the first three spherical harmonics Y_0^0, Y_1^0, and Y_1^1, as in Figure 12.5, are all spheres. Find their centers and radii.

12.2.20. Explain why the surface defined by $r = \widetilde{Y}_n^m(\varphi,\theta)$ is obtained by rotating that defined by $r = Y_n^m(\varphi,\theta)$ around the z–axis by $90°$.

12.2.21. Prove directly that the spherical Laplacian Δ_S is a self-adjoint linear operator with respect to the inner product (12.40).

12.2.22. (*a*) In view of Exercise 12.2.21, which orthogonality relations in (12.41) follow from their status as eigenfunctions of the spherical Laplacian?
(*b*) Prove the general orthogonality formulae by direct computation.

12.2.23. State and prove the orthogonality of the complex spherical harmonics (12.46). Then establish the following formula for their norms:

$$\| \mathcal{Y}_n^m \|^2 = \iint_{S_1} | \mathcal{Y}_n^m |^2 \, dS = \frac{4\pi (n+m)!}{(2n+1)(n-m)!} \qquad \begin{array}{l} n = 0, 1, 2, \ldots, \\ m = -n, -n+1, \ldots, n. \end{array} \tag{12.66}$$

12.2.24. Prove the formulae (12.42) for the norms of the spherical harmonics. *Hint*: Use Exercise 12.2.16.

12.2.25. Justify the formulas in (12.50) for (a) H_1^0, (b) H_2^0, (c) $\widetilde{H}_2^1$.

12.2.26. Find formulas for the following harmonic polynomials (*i*) in spherical coordinates; (*ii*) in rectangular coordinates: (a) H_4^0, (b) H_4^4, (c) $\widetilde{H}_4^4$.

12.2.27. Explain why every polynomial solution of the Laplace equation is a linear combination of the harmonic polynomials (12.49). *Hint*: Look at its Taylor series.

12.2.28. (a) Prove that if $u(x, y, z)$ is any harmonic polynomial, then so are $u(y, x, z)$, $u(z, x, y)$, and all other functions obtained by permuting the variables x, y, z. (b) Discuss the effect of such permutations on the basis harmonic polynomials $H_n^m(x, y, z)$ appearing in (12.50).

12.2.29. Find the formulas in rectangular coordinates for the following complementary harmonic functions: (a) K_0^0, (b) K_1^1, (c) K_2^0, (d) $\widetilde{K}_2^1$.

12.2.30. Let $u(x, y, z)$ be a harmonic function defined on the unit ball $r \leq 1$. Prove that its gradient at the center, $\nabla u(\mathbf{0})$, equals the average of the vector field $\mathbf{v}(\mathbf{x}) = \mathbf{x}\, u(\mathbf{x})$ over the unit sphere $r = 1$.

12.2.31. (a) Suppose $u(x, y, z)$ is a solution to the Laplace equation. Prove that the function $U(x, y, z) = r^{-1}\, u(x/r^2, y/r^2, z/r^2)$ obtained by *inversion* is also a solution. (b) Explain how inversion can be used to solve boundary value problems on the exterior of a sphere. (c) Use inversion to relate the solutions to Examples 12.6 and 12.7.

12.2.32. Suppose $u(r, \varphi, \theta)$ is the potential exterior to a spherical capacitor of unit radius. (a) Prove that $\lim_{r \to \infty} r\, u(r, \varphi, \theta)$ equals the average value of u on the sphere.
(b) Use Exercise 12.2.31 to deduce this result as a consequence of Theorem 12.4.

12.2.33. (a) Write out, using spherical coordinates, formulas for the L^2 inner product and norm for scalar fields $f(r, \varphi, \theta)$ and $g(r, \varphi, \theta)$ on a solid ball of unit radius centered at the origin.
(b) Let $f(x, y, z) = z$ and $g(x, y, z) = x^2 + y^2$. Find $\| f \|$, $\| g \|$ and $\langle f, g \rangle$.
(c) Verify the Cauchy–Schwarz and triangle inequalities for these two functions.

12.2.34. Use separation of variables to construct a Fourier series solution to the Laplace equation on a rectangular box, $B = \{0 < x < a,\ 0 < y < b,\ 0 < z < c\}$, subject to the Dirichlet boundary conditions $u(x,y,z) = \begin{cases} h(x,y), & z = 0,\ \ 0 < x < a,\ \ 0 < y < b, \\ 0, & \text{at all other points in } \partial B. \end{cases}$

12.2.35. Find the equilibrium temperature distribution inside a unit cube that has 100° temperature on its top face, 0° on its bottom face, while all four side faces are insulated.

12.2.36. Solve Exercise 12.2.35 when the top face of the cube has temperature $u(x,y,1) = \cos \pi x \, \cos \pi y$.

12.2.37. A solid unit cube is in thermal equilibrium when subject to 100° temperature on its top face and 0° on all other faces. *True or false*: The temperature at the center equals the average temperature over the surface of the cube.

12.2.38. Solve the boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + u = \cos x \cos y, \qquad 0 < x, y, z < \pi,$$

$$u(x,y,0) = 1, \quad \frac{\partial u}{\partial z}(x,y,\pi) = \frac{\partial u}{\partial y}(x,0,z) = \frac{\partial u}{\partial y}(x,\pi,z) = \frac{\partial u}{\partial z}(0,y,z) = \frac{\partial u}{\partial x}(\pi,y,z) = 0.$$

12.2.39. Let C be the cylinder of height 1 and diameter 1 that sits on the (x,y)–plane centered on the z–axis. (a) Write out, in cylindrical coordinates, the explicit formula for the L^2 inner product and norm on C.
(b) Let $f(x,y,z) = z$ and $g(x,y,z) = x^2 + y^2$. Find $\| f \|$, $\| g \|$ and $\langle f , g \rangle$.
(c) Verify the Cauchy–Schwarz and triangle inequalities for these two functions.

12.2.40. (a) Write out the Laplace equation in cylindrical coordinates.
(b) Use separation of variables to construct a series solution to the Laplace equation on the cylinder $C = \{x^2 + y^2 < 1,\ 0 < z < 1\}$, subject to the Dirichlet boundary conditions

$$u(x,y,z) = \begin{cases} h(x,y), & z = 0, \quad x^2 + y^2 < 1, \\ 0, & \text{at all other points in } \partial C. \end{cases}$$

12.2.41. A cylinder of radius 1 and height 2 has 100° temperature on its top face, 0° on its bottom face, while its curved side is fully insulated. Find its equilibrium temperature distribution.

12.2.42. Solve Exercise 12.2.41 if the curved sides are kept at 0° instead.

12.3 Green's Functions for the Poisson Equation

12.3.1. Find the equilibrium temperature of a sphere of radius 1 whose boundary is held at 0° while a concentrated unit heat source is applied at (*a*) the center; (*b*) a point half-way between the center and the boundary.

12.3.2. A hot soldering iron is continually applied to the north pole of a solid spherical ball of radius 1. Find the equilibrium temperature.

12.3.3. Write down the gravitional potential — both external and internal — due to a spherical planet of radius R composed out of a uniform material with density ρ.

12.3.4. (a) Find the gravitational potential due to a spherical shell of unit density obtained by carving out a spherical cavity of radius a from a solid ball of radius $b > a$. *Hint*: Use the solution to Exercise 12.3.3. (*b*) What is the gravitational force inside the cavity?
(*c*) Show that outside the shell, the gravitational potential is as if the entire mass were concentrated at the origin.

12.3.5. (*a*) Write down an integral formula for the gravitational potential and gravitational force field due to a mass of unit density in the shape of a solid unit cube that is centered at the origin. (*b*) Use numerical integration to determine the gravitational force vector at the points $(3, 0, 0)$ and $\left(\sqrt{3}, \sqrt{3}, \sqrt{3}\right)$. Before doing the calculation, see whether you can predict which experiences a stronger force, and then check your prediction numerically.
(*c*) Suppose the mass is re-formed into a sphere. How does this affect the gravitational force at the two points? First predict whether it will increase, decrease, or stay the same. Then test your prediction by computing the values and comparing with those you computed in part (*b*).

12.3.6. A thin hollow metal sphere of unit radius is grounded. Find the electrostatic potential inside the sphere due to a small solid metal ball of radius $\rho < 1$ placed at its center, assuming unit charge density throughout the ball.

12.3.7. A thin straight rod of unit density and length 2ℓ is fixed on the z–axis centered at the origin. Find the induced (*a*) gravitational potential and (*b*) gravitational force experienced by a point (x, y, z) not on the rod.

12.3.8. (*a*) Find the gravitational force due to a thin, uniform straight rod of unit density and infinite length by letting $\ell \to \infty$ in your solution to Exercise 12.3.7(*b*). (*b*) Show that the force field of part (a) has a potential function that can be identified with the two-dimensional logarithmic gravitational potential due to a point mass at the origin. Thus, two-dimensional gravitation can be regarded as a cross-section of three-dimensional gravitation due to infinitely long vertical line masses. (*c*) Is your potential function the limit, as $\ell \to \infty$, of the potential function you found in Exercise 12.3.7(a)? Discuss.

12.3.9. Which well-known solutions to the Laplace equation comes from setting $m = n = 0$ in (12.61)?

12.3.10. Use the Fredholm Alternative to analyze the existence and uniqueness of solutions to the homogeneous Neumann boundary value problem for the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^3$.

12.3.11. Mimic the proof of Theorem 6.19 to establish the solution formula (12.72).

12.3.12. Use the Method of Images to find the Green's function for a solid hemisphere of unit radius subject to homogeneous Dirichlet boundary conditions.

12.4 The Heat Equation for the Three-Dimensional Media

12.4.1. Let $B = \{ 0 < x < a,\ 0 < y < b,\ 0 < z < c \}$ be a solid box of size $a \times b \times c$. (*a*) Write down an initial-boundary value problem for the thermodynamics of the box when all its sides are all held at 0° and its initial temperature is $f(x, y, z)$. (*b*) Use separation of variables to construct the normal mode solutions. (*c*) Write down a series representing the general solution to the initial-boundary value problem. What are the formulas for the coefficients in your series? (*d*) What is the equilibrium temperature? How fast does the temperature in the box decay to equilibrium?

12.4.2. *True or false*: In the context of Exercise 12.4.1, among all boxes of a given volume V, a cube decays slowest to thermal equilibrium. What is the cube's decay rate?

12.4.3. Answer Exercises 12.4.1 and 12.4.2 when the top of the box, where $z = c$, is insulated.

12.4.4. A rectangular brick of size 1 cm $\times$ 2 cm $\times$ 3 cm made out of material with diffusion coefficient $\gamma = 6$ is insulated on five sides, while one of its small ends is held at temperature $u(x, y, 0) = \cos \pi x \, \cos 2\pi y$. (a) Find the eventual equilibrium temperature distribution. (*b*) If the brick is initially heated in an oven, how fast does it return to equilibrium?

12.4.5. Let $C = \left\{ 0 \le \sqrt{x^2 + y^2} < a,\ 0 < z < h \right\}$ be a solid cylinder of radius a and height h.

(*a*) Write down an initial-boundary value problem in cylindrical coordinates for the thermodynamics of the cylinder when its sides, top, and bottom are all held at 0°.

(*b*) Use separation of variables to write down a series representing the general solution to the initial-boundary value problem. What are the formulas for the coefficients in your series?

(*c*) What is the eventual equilibrium temperature?

(*d*) How fast does the temperature in the cylinder go to equilibrium?

12.4.6. Find the solution to the initial-boundary value problem in Exercise 12.4.5 when the initial temperature of the cylinder is uniformly 30°. *Hint*: Use (11.112) to evaluate the coefficients.

12.4.7. A cylindrical can that contains 355 ml of soda is removed from the refrigerator. Find the optimal cylindrical shape for such a can in order to keep the soda cold the longest. Is this the manufactured shape of a standard soda can?

12.4.8. *True or false*: Among all solid cylinders of a given volume, the one that reaches thermal equilibrium the slowest, when subject to homogeneous Dirichlet boundary conditions, is the one that has the least surface area. Justify your answer.

12.4.9. Among all fully insulated solid cylinders of unit volume, which cools down
(i) the slowest? (ii) the fastest?

12.4.10. Write down a series for the solution to the homogeneous Neumann boundary value problem for the heat equation on a bounded domain $\Omega \subset \mathbb{R}^3$, corresponding to the thermodynamics of a completely insulated solid body. What is the equilibrium temperature of the body? Does the solution decay to equilibrium? If so, how fast?

12.4.11. Suppose $u(t,x,y,z)$ is a solution to the heat equation on a fully insulated bounded domain $\Omega \subset \mathbb{R}^3$. Use the identities in Exercise 12.1.11 to prove the following:

(a) The total heat $H(t) = \iiint_\Omega u(t,x,y,z)\,dx\,dy\,dz$ is conserved, i.e., is constant. Explain how this can be used to determine the equilibrium temperature of the body.

(b) If u is a non-equilibrium solution, its squared L^2 norm $E(t) = \iiint_\Omega u(t,x,y,z)^2\,dx\,dy\,dz$ is a strictly decreasing function of t.

(c) Use part (b) to prove uniqueness of solutions to the initial value problem.

12.4.12. State and prove a Maximum Principle for the three-dimensional heat equation.

12.4.13. It takes a solid ball of radius 1 cm ten minutes to return to (approximate) thermal equilibrium. How long does it take a similar ball of radius 2?

12.4.14. If a 200-gram potato served hot from the oven takes 15 minutes until its maximum temperature is less than 40° C, how long does it take a 300-gram potato of the same shape to cool off?

12.4.15. A uniform solid metal ball of radius 1 meter, with diffusion coefficient $\gamma = 2$, is taken from a 300° oven and immersed in a bucket of ice water. (a) Write down an initial-boundary value problem that describes the temperature of the ball. (b) Find a series solution for the temperature. (c) At what time is the temperature $\le 50^\circ$ throughout the ball?

12.4.16. Find the decay rate to thermal equilibrium of a solid spherical ball of radius R and diffusion coefficient γ when subject to homogeneous Dirichlet boundary conditions.

12.4.17. *True or false*: A heated solid hemisphere placed in a 0° environment cools down twice as fast as a solid sphere of the same radius made out of the same material.

12.4.18. A fully insulated solid spherical ball of radius 1 has initial temperature distribution $f(r,\varphi,\theta)$. (*a*) Write down a formula for the equilibrium temperature of the ball. (*b*) What is the rate of decay of the ball to thermal equilibrium?

12.4.19. Which cools down to equilibrium faster: a fully insulated solid ball or one whose boundary is held fixed at 0°? How much faster?

12.4.20. A solid sphere and solid cube are made out of the same material and have the same volume. Both are heated in an oven and then submerged in a large vat of water. Which will cool down faster? Explain and justify your answer.

12.4.21. Answer Exercise 12.4.20 when the two solids have the same surface area.

12.4.22. Suppose the solid spherical shell in Exercise 12.2.7 starts off at room temperature. Assuming that the water in the center remains at 100°, find the rate at which the shell tends to thermal equilibrium.

12.4.23. The thermodynamics of a thin, uniform, spherical shell of unit radius is governed by the *spherical heat equation* $u_t = \gamma \Delta_S u$, $u(0,\varphi,\theta) = f(\varphi,\theta)$, in which Δ_S is the spherical Laplacian (12.19). The solution $u(t,\varphi,\theta)$ represents the temperature of the point on the unit sphere with angular coordinates φ,θ, while $f(\varphi,\theta)$ is the initial temperature distribution. (*a*) Find the eigensolutions. (*b*) Write down the solution to the initial value problem as a series in eigensolutions. (*c*) What is the final equilibrium temperature of the spherical shell? (*d*) What is its rate of decay to equilibrium? (*e*) Find the solution and the final equilibrium temperature when $f(\varphi,\theta) =$ (*i*) $\sin\varphi\cos\theta$; (*ii*) $\cos 2\varphi$.

12.4.24. A spherical potato, of radius $R = 7.5\,\text{cm}$ and thermal diffusivity $\gamma = .3\,\text{cm}^2/\text{sec}$, is initially at room temperature, 25° C, and is placed in a pot of boiling water at 100° C. The potato is cooked when it has reached the temperature of at least 90° C throughout. How long do you have to wait until the potato is done?

12.4.25. (*a*) Explain why the spherical Bessel function $S_1(x)$ is bounded at $x = 0$. What is $S_1(0)$? (*b*) Answer the same question for $S_2(x)$.

12.4.26. Prove the formulae (12.108).

12.4.27. (a) Find a recurrence relation expressing the spherical Bessel function $S_{m-1}(x)$ in terms of $S_m(x)$. (b) Prove that

$$\frac{d}{dx}\left[x^3 \big(S_m(x)^2 - S_{m-1}(x)\, S_{m+1}(x) \big) \right] = 2x^2\, S_m(x)^2.$$

12.4.28. Let $m \geq 0$ be a fixed integer. (a) Prove that the rescaled spherical Bessel functions $v_n(r) = S_m(\sigma_{m,n}\, r)$, $n = 1, 2, \ldots$, are mutually orthogonal under the inner product $\langle f, g \rangle = \int_0^1 f(r)\, g(r)\, r^2\, dr$. (b) Prove that $\| v_n \| = \frac{1}{\sqrt{2}} \,|\, S_{m+1}(\sigma_{m,n}) \,|$. *Hint*: Mimic the method outlined in Exercise 11.4.22, using the identity in Exercise 12.4.27(b).

12.4.29. (a) Use the result of Exercise 12.4.28 to prove the formulae (12.116) for the L^2 norms of the eigenfunctions (12.110). (b) Justify the formulae (12.115).

12.4.30. *True or false*: In a three-dimensional medium, heat energy propagates at infinite speed.

12.4.31. A solid spherical ball of radius 1 is heated to 100° and inserted into a three-dimensional medium filling the rest of $\mathbb{R}^3$ with uniform temperature 0°.
(a) Write down an integral formula for the subsequent temperature distribution over $\mathbb{R}^3$ at time $t > 0$, assuming a common diffusion coefficient $\gamma = 1$.
(b) Evaluate the resulting integral using spherical coordinates.

12.4.32. (a) Prove that $u(t, r)$ is a spherically symmetric solution to the three-dimensional heat equation if and only if $w(t, r) = r\, u(t, r)$ solves the one-dimensional heat equation: $w_t = w_{rr}$. (b) *True or false*: If $w(t, r)$ is the fundamental solution for the one-dimensional heat equation based at $r = 0$, then $u(t, r) = w(t, r)/r$ is the fundamental solution for the three-dimensional heat equation based at the origin.

12.4.33. Construct the solution to the initial value problem in Exercise 12.4.31 using radial symmetry and Exercise 12.4.32.

12.4.34. Suppose that, as Earth orbits the sun, its surface is subject to yearly periodic temperature variations $a \cos \omega\, t$, where the frequency ω is given by (4.56). (a) Assuming, for simplicity, that the Earth is a homogeneous solid ball, of radius R, formulate an initial-boundary value problem that governs the temperature fluctuations within the Earth due to its orbiting the sun. (b) At what depth does the temperature vary out of phase with the surface, i.e., is the warmest in winter and coldest in summer? Compare your answer with the root cellar computation at the end of Section 4.1. *Hint*: Use Exercise 12.4.32.

12.4.35. (a) Prove that if $u(t, x)$ is any (sufficiently smooth) solution to the heat equation, so is its time derivative $v = \partial u / \partial t$. (b) Write out the time derivative of the fundamental solution, and the initial value problem it satisfies.

12.4.36. Write down an explicit eigenfunction series for the fundamental solution $F(t, \mathbf{x}; \boldsymbol{\xi})$ to the heat equation in a unit cube with thermal diffusivity $\gamma = 1$ that is subject to homogeneous Dirichlet boundary conditions.

12.4.37. Write down an explicit eigenfunction series for the fundamental solution $F(t, \mathbf{x}; \boldsymbol{\xi})$ to the heat equation in a ball of radius 1 that has thermal diffusivity $\gamma = 1$ and is subject to homogeneous Dirichlet boundary conditions.

12.4.38. Justify the statement that formula (12.119) provides a solution to the three-dimensional heat equation.

12.4.39. Fill in the details of the proof of Theorem 12.14.

12.5 The Wave Equation for Three-Dimensional Media

12.5.1. Find the eigenfunction series solution to the initial-boundary value problem for the wave equation $u_{tt} = \Delta u$ on a unit cube $C = \{0 < x, y, z < 1\}$, subject to homogeneous Dirichlet boundary conditions and one of the following sets of initial conditions: (*a*) $u(0, x, y, z) = 1$, $u_t(0, x, y, z) = 0$; (*b*) $u(0, x, y, z) = 0$, $u_t(0, x, y, z) = 1$; (*c*) $u(0, x, y, z) = \sin \pi x \, \sin \pi y \, \sin \pi z$, $u_t(0, x, y, z) = 0$; (*d*) $u(0, x, y, z) = \sin 3\pi x$, $u_t(0, x, y, z) = \sin 2\pi y$; (*e*) $u(0, x, y, z) = 0$, $u_t(0, x, y, z) = x\,y\,z\,(1 - x)(1 - y)(1 - z)$.

12.5.2. Suppose the cube in Exercise 12.5.1 is subject to homogeneous Neumann boundary conditions. Which of the preceding initial value problems leads to an unstable motion of the cube?

12.5.3. (*a*) Find the separable periodic vibrations of a unit cube subject to homogeneous Dirichlet boundary conditions. (*b*) Can you find a periodic mode that is not separable?

12.5.4. Answer Exercise 12.5.3 when one face of the cube is left free, while the other five faces are fixed.

12.5.5. Given a material with wave speed $c = 1.5$ cm/sec, find the natural vibrational frequencies of a solid rectangular box of size 1 cm $\times$ 2 cm $\times$ 3 cm whose sides are held fixed. List the lowest five such frequencies in order. Does the box vibrate periodically?

12.5.6. Find the natural vibrational frequencies of a solid cylinder of height 2, radius 1, and wave speed $c = 1$, when (*a*) all sides are fixed; (*b*) the top and bottom of the cylinder are free, while the curved side is fixed; (*c*) the curved side of the cylinder is free, while the top and bottom are fixed.

12.5.7. Among all solid cylinders of unit volume with fixed boundary, find the one that vibrates the slowest.

12.5.8. Does a solid spherical ball that is subject to homogeneous Neumann boundary conditions vibrate (*i*) faster, (*ii*) slower, or (*iii*) at the same rate as the same ball subject to homogeneous Dirichlet conditions. If your answer is (*i*) or (*ii*), estimate how much faster or slower.

12.5.9. A solid cube and solid sphere are made of the same material and have the same volume. Which vibrates faster when subject to homogeneous Dirichlet boundary conditions?

12.5.10. Assuming that they both have the same wave speed and fixed boundaries, which vibrates faster: a solid sphere or a circular membrane of the same radius?

12.5.11. A uniform, solid spherical planet is floating freely in outer space. Find its three slowest resonant frequencies.

12.5.12. *True or false*: Suppose we have two uniform solid bodies composed of the same material. If the first body cools down to thermal equilibrium the fastest, then it also vibrates the fastest. Explain your answer.

12.5.13. (a) Define what is meant by a nodal curve and a nodal region on a vibrating thin spherical shell. (*b*) *True or false*: All the nodal curves are arcs of circles.

12.5.14. The propagation of electromagnetic waves (including light) is governed by the electric field $\mathbf{E}(t, \mathbf{x})$ and magnetic field $\mathbf{B}(t, \mathbf{x})$, which are both time-dependent vector fields defined for $\mathbf{x} = (x, y, z)$ in a domain $\Omega \subset \mathbb{R}^3$. In empty space, *Maxwell's equations* (as formulated by Heaviside) are

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu_0\, \epsilon_0} \nabla \times \mathbf{B}, \tag{12.136}$$

where μ_0, ϵ_0 are, respectively, the *permeability* and *permittivity* constants. Prove that all individual components of $\mathbf{E}$ and $\mathbf{B}$ satisfy the scalar wave equation. What is the wave speed, i.e., the speed of light in empty space?

12.6 Spherical Waves and Huygen's Principle

12.6.1. Solve the wave equation in three-dimensional space for the following initial conditions: (*a*) $u(0,x,y,z) = x + z$, $u_t(0,x,y,z) = 0$; (*b*) $u(0,x,y,z) = 0$, $u_t(0,x,y,z) = y$; (*c*) $u(0,x,y,z) = 1/(1 + x^2 + y^2 + z^2)$, $u_t(0,x,y,z) = 0$, (*d*) $u(0,x,y,z) = 0$, $u_t(0,x,y,z) = 1/(1 + x^2 + y^2 + z^2)$.

12.6.2. At what points in space-time does a three-dimensional wave vanish if it vanishes outside a sphere of radius R at the initial time $t = 0$?

12.6.3. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \qquad u(0,x,y,z) = 0, \qquad \frac{\partial u}{\partial t}(0,x,y,z) = \begin{cases} 1, & 0 < x,y,z < 1, \\ 0, & \text{otherwise}, \end{cases}$$

i.e., the initial velocity is 1 inside a unit cube and 0 outside the cube. We interpret the solution $u(t,x,y,z)$ as the intensity of light at a given point in space-time, measured in units that make the speed of light $c = 1$. (a) Write down an integral formula for $u(t,x,y,z)$. (b) Suppose a light sensor is placed at the point $(2,2,1)$. For which values of $t > 0$ will the sensor register a nonzero signal? Sketch a rough graph of what the sensor measures. (You do not need to find the precise formula, but explain how you obtained your graph.) (c) *True or false*: The solution $u(t,x,y,z) \geq 0$ at all points in space-time.

12.6.4. Is (12.151) a solution to the wave equation for $t < 0$? If not, write down a solution formula that is valid for negative t.

12.6.5. *True or false*: The function $u(t,x,y,z)$ defined by (12.154) is everywhere continuous.

12.6.6. A thermonuclear explosion occurs at the center of the Earth. Would you feel the effect first through a motion at the surface or a change in temperature at the surface? Discuss.

12.6.7. Prove that the area of the spherical cap $S_t^{\mathbf{x}} \cap B_1$ is given by formula (12.152).

12.6.8. Solve initial value problem for the two-dimensional wave equation with the following initial data (a) $u(0,x,y) = x - y$, $u_t(0,x,y) = 0$; (b) $u(0,x,y) = 0$, $u_t(0,x,y) = y$.

12.6.9. (a) Prove that $u(t,x,y) = 1/\sqrt{x^2+y^2-c^2t^2}$ is a solution to the two-dimensional wave equation on the domain $\Omega = \{x^2+y^2 > c^2t^2\}$ exterior to the light cone passing through the origin. What is the corresponding initial data at $t = 0$? (b) Use part (a) to solve the initial value problem $u(0,x,y) = 0$, $u_t(0,x,y) = 1/\sqrt{x^2+y^2}$, on Ω.

12.6.10. Consider the two-dimensional wave equation on $\mathbb{R}^2$ with wave speed $c = 1$. Write down an integral formula for the solution to the following initial value problems. You need not evaluate the integrals. (a) $u(0,x,y) = x^3 - y^3$, $u_t(0,x,y) = 0$;
(b) $u(0,x,y) = 0$, $u_t(0,x,y) = y^2$; (c) $u(0,x,y) = x^2 + y^2$, $u_t(0,x,y) = -x^2 - y^2$.

12.6.11. (a) Find the solution to the two-dimensional wave equation whose initial displacement is a concentrated delta impulse at the origin and whose initial velocity is zero.
(b) Is your expression a classical solution when $t > 0$?
(c) *True or false*: The solution tends to 0 uniformly as $t \to \infty$.

12.6.12. Use separation of variables to write down an eigenfunction series solution to the partial differential equation (12.169) when subject to homogeneous Dirichlet boundary conditions at $r = 1$ and bounded at $r = 0$.

12.6.13. Write down the fundamental solution for the one-dimensional wave equation with (a) a concentrated initial displacement at the origin; (b) a concentrated initial velocity at the origin. (c) Discuss the validity of Huygens' Principle in a one-dimensional universe.

12.6.14. Discuss how you can construct solutions to the one-dimensional wave equation by descent from the three-dimensional wave equation.

12.7 The Hydrogen Atom

12.7.1. If the nucleus contains Z protons circled by a single electron, then its atomic potential $V(\mathbf{x})$ is rescaled accordingly, replacing α^2/r by $Z\alpha^2/r$. Discuss the induced effect on the energy levels of such an atomic ion.

12.7.2. (a) Write down the time-dependent wave function for a single electron atom when the electron is in its ground state, i.e., the lowest energy level. (*b*) What is the probability density of the electron? (*c*) What is the probability of finding the electron within 1 Bohr radius of the atom? (*d*) Find the distance d (measured in Bohr radii) so that there is a 95% probability of finding the electron within a distance d of the nucleus.

12.7.3. Prove that the two expressions for the Laguerre polynomials in (12.186) agree.

12.7.4. (a) Let $k = 0, 1, 2, \ldots$ be a nonnegative integer. The *Laguerre differential equation* of *order* k is

$$x\,u'' + (1-x)\,u' + k\,u = 0. \tag{12.191}$$

Show that $x = 0$ is a regular singular point. Then prove that the Frobenius solution based at $x = 0$ is a polynomial of degree j that coincides with the Laguerre polynomial $L_k^0(x)$.
(*b*) Given nonnegative integers $j, k \geq 0$, use the Frobenius method to prove that the *generalized Laguerre differential equation*

$$x\,u'' + (j+1-x)\,u' + k\,u = 0 \tag{12.192}$$

has a polynomial solution that can be identified with the generalized Laguerre polynomial $L_k^j(x)$ in (12.186).

12.7.5. Suppose that $P(s)$ solves the ordinary differential equation (12.182). Prove that $Q(s) = s^{-l}\,e^{s/2}\,P(s)$ solves the differential equation

$$s\,\frac{d^2Q}{ds^2} + [2(l+1) - s]\,\frac{dQ}{ds} + (n-l-1)Q = 0. \tag{12.193}$$

Then apply the result of Exercise 12.7.4 to complete the proof of Theorem 12.19.

12.7.6. Suppose $f(x)$ is a polynomial, and let $L_k^j(s)$ denote the generalized Laguerre polynomials (12.186). (a) Prove that, for $j, k \geq 0$,

$$\int_0^\infty f(s)\,L_k^j(s)\,s^j\,e^{-s}\,ds = \frac{(-1)^k}{k!}\int_0^\infty f^{(k)}(s)\,s^{j+k}\,e^{-s}\,ds.$$

(*b*) For fixed j, prove that the generalized Laguerre polynomials $L_k^j(s)$, $k = 0, 1, 2, \ldots,$ are orthogonal with respect to the weighted inner product $\langle f, g\rangle = \int_0^\infty f(s)\,g(s)\,s^j\,e^{-s}\,ds$.

(*c*) Prove the formula for their corresponding norms: $\|\,L_k^j\,\| = \sqrt{\dfrac{(j+k)!}{k!}}$.

12.7.7. (a) Prove that the generalized Laguerre polynomials satisfy the following recurrence relation:

$$(k+1)\,L^j_{k+1}(s) - (j+2k+1-s)\,L^j_k(s) + (j+k)\,L^j_{k-1}(s) = 0. \tag{12.194}$$

(b) Prove that

$$\int_0^\infty s^{j+1} e^{-s} \left[L^j_k(s)\right]^2 ds = \frac{(j+2k+1)\,(j+k)!}{k!}. \tag{12.195}$$

Hint: Use part (a) and Exercise 12.7.6.

12.7.8. Prove that the atomic eigenfunctions (12.188) form an orthonormal system of wave functions with respect to the L^2 inner product on $\mathbb{R}^3$. *Hint*: Use Theorem 9.33 and equation (12.195).