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Ch11 Dynamics of Planar Media

11.1 Diffusion in Planar Media

11.1.1. A homogeneous, isotropic circular metal disk of radius 1 meter has its entire boundary insulated. The initial temperature at a point is equal to the distance of the point from the center. Formulate an initial-boundary value problem governing the disk's subsequent temperature dynamics. What is the eventual equilibrium temperature of the disk?

11.1.2. A homogeneous, isotropic, circular metal disk of radius 2 cm has half its boundary fixed at 100° and the other half insulated. Given a prescribed initial temperature distribution, set up the initial-boundary value problem governing its subsequent temperature profile. What is the eventual equilibrium temperature of the disk? Does your answer depend on the initial temperature?

11.1.3. Given the initial temperature distribution $f(x,y) = x\,y\,(1-x)(1-y)$ on the unit square $\Omega = \{0 \le x, y \le 1\}$, determine the equilibrium temperature when subject to homogeneous (*a*) Dirichlet boundary conditions; (*b*) Neumann boundary conditions.

11.1.4. A square plate with side lengths 1 meter has its right and left edges insulated, its top edge held at 100°, and its bottom edge held at 0°. Assuming that the plate is made out of a homogeneous, isotropic material, formulate an appropriate initial-boundary value problem describing the temperature dynamics of the plate. Then find its eventual equilibrium temperature.

11.1.5. A square plate with side lengths 1 meter has initial temperature 5° throughout, and evolves subject to the Neumann boundary conditions $\partial u / \partial \mathbf{n} = 1$ on its entire boundary. What is the eventual equilibrium temperature?

11.1.6. Let $u(t,x,y)$ be a solution to the heat equation on a bounded domain Ω subject to homogeneous Neumann conditions on its boundary $\partial\Omega$. (*a*) Prove that the total heat $H(t) = \iint_\Omega u(t,x,y)\,dx\,dy$ is conserved, i.e., is constant in time. (*b*) Use part (*a*) to prove that the eventual equilibrium solution is everywhere equal to the average of the initial temperature $u(0,x,y)$. (*c*) What can you say about the behavior of the total heat for the homogeneous Dirichlet boundary value problem? (*d*) What about an inhomogeneous Dirichlet boundary value problem?

11.1.7. Let $u(t,x,y)$ be a nonconstant solution to the heat equation on a connected, bounded domain Ω subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. (*a*) Prove that its L^2 norm $N(t) = \sqrt{\iint_\Omega u(t,x,y)^2\,dx\,dy}$ is a strictly decreasing function of t. (*b*) Is this also true for mixed boundary conditions? (*c*) For Neumann boundary conditions?

11.1.8. Are the conclusions in Exercises 11.1.6 and 11.1.7 valid for the general diffusion equation (11.12)?

11.1.9. Write out the eigenvalue equation governing the separable solutions to the general diffusion equation (11.11), subject to appropriate boundary conditions. Given a complete system of eigenfunctions, write down the eigenfunction series solution to the initial value problem $u(0,x,y) = f(x,y)$, including the formulas for the coefficients.

11.1.10. *True or false*: The equilibrium temperature of a fully insulated nonuniform plate whose thermodynamics are governed by the general diffusion equation (11.12) equals the average initial temperature.

11.1.11. Let $\alpha > 0$, and consider the initial-boundary value problem $u_t = \Delta u - \alpha\, u$, $u(0,x,y) = f(x,y)$ on a bounded domain $\Omega \subset \mathbb{R}^2$, with boundary conditions $\partial u/\partial \mathbf{n} = 0$ on $\partial\Omega$.
(*a*) Write the equation in self-adjoint form (9.122). *Hint*: Look at Exercise 9.3.26.
(*b*) Prove that the problem has a unique equilibrium solution.

11.1.12. Write each of the following linear evolution equations in the self-adjoint form (9.122) by choosing suitable inner products and a suitable set of homogeneous boundary conditions. Is the operator you construct positive definite?

(*a*) $u_t = u_{xx} + u_{yy} - u$, (*b*) $u_t = y\,u_{xx} + x\,u_{yy}$, (*c*) $u_t = \Delta^2 u$.

11.1.13. Prove that if $f(x,y)$ is continuous and $\displaystyle\iint_R f(x,y)\,dx\,dy = 0$ for all $R \subset \Omega$, then $f(x,y) \equiv 0$ for $(x,y) \in \Omega$. *Hint*: Adapt the method in Exercise 6.1.23.

11.2 Explicit Solution of the Heat Equation

11.2.1. A rectangle of size 2 cm by 1 cm has initial temperature $f(x,y) = \sin \pi x \, \sin \pi y$ for $0 \le x \le 2$, $0 \le y \le 1$. All four sides of the rectangle are held at 0°. Assuming that the thermal diffusivity of the plate is $\gamma = 1$, write down a formula for its subsequent temperature $u(t,x,y)$. What is the rate of decay to thermal equilibrium?

11.2.2. Solve Exercise 11.2.1 when the initial temperature $f(x,y)$ is

(*a*) $x\,y$, (*b*) $\begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2; \end{cases}$ (*c*) $\bigl(1 - |\,1 - x\,|\bigr)\bigl(\tfrac{1}{2} - \bigl|\,\tfrac{1}{2} - y\,\bigr|\bigr)$.

11.2.3. Solve the initial-boundary value problem for the heat equation $u_t = 2\,\Delta u$ on the rectangle $-1 < x < 1$, $0 < y < 1$ when the two short sides are kept at 0°, the two long sides are insulated, and the initial temperature distribution is $u(0,x,y) = \begin{cases} -1, & x < 0, \\ +1, & x > 0, \end{cases}$ $0 < y < 1$.

11.2.4. Answer Exercise 11.2.3 when the two long sides are kept at 0° and the two short sides are insulated.

11.2.5. A rectangular plate of size 1 meter by 3 meters is made out a metal with unit diffusivity. The plate is taken from a 0° freezer, and, from then on, one of its long sides is heated to 100°, the other is held at 0°, while its top, bottom, and both of the short sides are fully insulated. (a) Set up the initial-boundary value problem governing the time-dependent temperature of the plate. (b) What is the equilibrium temperature? (c) Use your answer from part (b) to construct an eigenfunction series for the solution. (d) How long until the temperature of the plate is everywhere within 1° of its eventual equilibrium?
Hint: Once t is no longer small, you can approximate the series solution by its first term.

11.2.6. Among all rectangular plates of a prescribed area, which one returns to thermal equilibrium the slowest when subject to Dirichlet boundary conditions? The fastest? Use your physical intuition to explain your answer, but justify it mathematically.

11.2.7. Answer Exercise 11.2.6 for a fully insulated rectangular plate, i.e., subject to Neumann boundary conditions.

11.2.8. A square metal plate is taken from an oven, and then set out to cool, with its top and bottom insulated. Find the rate of cooling, in terms of the side length and the thermal diffusivity, if (a) all four sides are held at 0°; (b) one side is insulated and the other three sides are held at 0°; (c) two adjacent sides are insulated and the other two are held at 0°; (d) two opposite sides are insulated and the other two are held at 0°; (e) three sides are insulated and the remaining side is held at 0°. Order the cooling rates of the plates from fastest to slowest. Do your results confirm your intuition?

11.2.9. Two square plates are made out of the same homogeneous material, and both are initially heated to 100°. All four sides of the first plate are held at 0°, whereas one of the sides of the second plate is insulated while the other three sides are held at 0°. Which plate cools down the fastest? How much faster? Assuming the thermal diffusivity $\gamma = 1$, how long do you have to wait until every point on each plate is within 1° of its equilibrium temperature? *Hint*: Once t is no longer small, the series solution is well approximated by its first term.

11.2.10. *Multiple choice*: On a unit square that is subject to Dirichlet boundary conditions, the eigenvalues of the Laplace operator are
(a) all simple, (b) at most double, or (c) can have arbitrarily large multiplicity.

11.2.11. The thermodynamics of a thin circular cylindrical shell of radius a and height h, e.g., the side of a tin can after its top and bottom are removed, is modeled by the heat equation $\dfrac{\partial u}{\partial t} = \gamma \left(\dfrac{1}{a^2} \dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2} \right)$, in which $u(t, \theta, z)$ measures the temperature of the point on the cylinder at time $t > 0$, angle $-\pi < \theta \leq \pi$, and height $0 < z < h$. Keep in mind that $u(t, \theta, z)$ must be a 2π–periodic function of the angular coordinate θ. Assume that the cylinder is everywhere insulated, while its two circular ends at held at 0°. Given an initial temperature distribution at time $t = 0$, write down a series formula for the cylinder's temperature at subsequent times. What is the eventual equilibrium temperature? How fast does the cylinder return to equilibrium?

11.2.12. Consider the initial-boundary value problem

$$u_t = u_{xx} + u_{yy}, \qquad u(0,x,y) = 0, \qquad 0 < x, y < \pi, \qquad t > 0,$$

for the heat equation in a square subject to the Dirichlet conditions

$$u(0,y) = u(\pi,y) = 0 = u(x,0), \qquad u(x,\pi) = f(x), \qquad 0 < x, y < \pi.$$

Write out an eigenfunction series formulas for
(*a*) the equilibrium solution $u_\star(x,y) = \lim_{t \to \infty} u(t,x,y)$; (*b*) the solution $u(t,x,y)$.

11.2.13. Solve Exercise 11.2.1 when one long side of the plate is held at 100°.
Hint: See Exercise 11.2.12.

Heating of a Disk

11.3 Series Solutions of ODE

The Gamma Function

The Airy Equation

11.3.1. Find (*a*) $\Gamma\left(\frac{5}{2}\right)$, (*b*) $\Gamma\left(\frac{7}{2}\right)$, (*c*) $\Gamma\left(-\frac{3}{2}\right)$, (*d*) $\Gamma\left(-\frac{5}{2}\right)$.

11.3.2. Prove that $\Gamma\left(n + \frac{1}{2}\right) = \dfrac{\sqrt{\pi}\,(2n)!}{2^{2n}\, n!}$ for every positive integer n.

11.3.3. Let $x \in \mathbb{C}$ be complex. (*a*) Prove that the gamma function integral (11.62) converges, provided $\operatorname{Re} x > 0$. (*b*) Is formula (11.63) valid when x is complex?

11.3.4. Prove that $\Gamma(x) = \int_0^1 (-\log s)^{x-1}\, ds$, and hence, for $0 \leq n \in \mathbb{Z}$, we have $n! = \int_0^1 (-\log s)^n\, ds$. *Remark*: Euler first established the latter identity directly, and used it to define the gamma function.

11.3.5. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$.

11.3.6. Can you construct a function $f(x)$ that satisfies the factorial functional equation (11.61) and has the values $f(x) = 1$ for $0 \leq x \leq 1$? If so, is $f(x) = \Gamma(x+1)$?

11.3.7. Explain how to construct the power series for $\sin x$ by solving the differential equation (11.67).

11.3.8. Construct two independent power series solutions to the Euler equation $x^2 u'' - 2u = 0$ based at the point $x_0 = 1$.

11.3.9. Construct two independent power series solutions to the equation $u'' + x^2 u = 0$ based at the point $x_0 = 0$.

11.3.10. Consider the ordinary differential equation $u'' + 2x u' + 2u = 0$. (a) Find two linearly independent power series solutions in powers of x. (b) What is the radius of convergence of your power series? (c) By inspection of your series, find one solution to the equation expressible in terms of elementary functions. (d) Find an explicit (non-series) formula for the second independent power series solution.

11.3.11. Answer Exercise 11.3.10 for the equation $u'' + \frac{1}{2} x u' - \frac{1}{2} u = 0$, which is a special case of equation (8.63).

11.3.12. Consider the ordinary differential equation $u'' + x u' + 2u = 0$. (a) Find two linearly independent power series solutions based at $x_0 = 0$. (b) Write down the power series for the solution to the initial value problem $u(0) = 1$, $u'(0) = -1$. (c) What is the radius of convergence of your power series solution in part (a)? Can you justify this by direct inspection of your power series?

11.3.13. The *Hermite equation* of order n is

$$\frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + 2n u = 0. \tag{11.85}$$

Assuming $n \in \mathbb{N}$ is a nonnegative integer: (a) Find two linearly independent power series solutions based at $x_0 = 0$, and then show that one of your solutions is a polynomial of degree n. (b) Prove that the Hermite polynomial $H_n(x)$ defined in (8.64) solves the Hermite equation (11.85) and hence is a multiple of the polynomial solution you found in part (a). What is the multiple? (c) Prove that the Hermite polynomials are orthogonal with respect to the inner product $\langle u , v \rangle = \int_{-\infty}^{\infty} u(x)\, v(x)\, e^{-x^2}\, dx$.

11.3.14. Use the ratio test to directly determine the radius of convergence of the series solutions (11.81, 82) to the Airy equation.

11.3.15. Write down the general solution to the following ordinary differential equations:
(a) $u'' + (x - c)\, u = 0$, where c is a fixed constant;
(b) $u'' = \lambda x u$, where $\lambda \neq 0$ is a fixed nonzero constant.

11.3.16. The *Airy function of the second kind* is defined by

$$\mathrm{Bi}(x) = \frac{1}{\pi} \int_0^{\infty} \left[\exp\left(s x - \tfrac{1}{3} s^3 \right) + \sin\left(s x + \tfrac{1}{3} s^3 \right) \right] ds. \tag{11.86}$$

(a) Prove that $\mathrm{Bi}(x)$ is well defined and a solution to the Airy equation. (b) Given that†

$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\,\Gamma\left(\frac{2}{3}\right)}, \qquad \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}, \tag{11.87}$$

explain why every solution to the Airy equation can be written as a linear combination of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. (c) Write the two series solutions (11.81, 82) in terms of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.

11.3.17. Use the Fourier transform to construct an L^2 solution to the Airy equation. Can you identify your solution?

11.3.18. Apply separation of variables to the Tricomi equation (4.137), and write down all separable solutions. *Hint*: See Exercise 11.3.15(*b*) and Exercise 11.3.16.

11.3.19. (*a*) Show that $u(x) = \sum_{n=1}^{\infty} (n-1)!\, x^n$ is a power series solution to the first-order linear ordinary differential equation $x^2 u' - u + x = 0$. (*b*) For which x does the series converge? (*c*) Find an analytic formula for the general solution to the equation. (*d*) Find a second-order homogeneous linear ordinary differential equation that has this power series as a (formal) solution. *Remark*: The lesson of this exercise is that not all power series solutions to ordinary differential equations converge. Theorem 11.5 guarantees convergence at a regular point, but in this example the power series is based at the singular point $x_0 = 0$.

11.3.20. *True or false*: The only function $f(x)$ that has identically zero Taylor series is the zero function.

11.3.21. Define $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ (a) Prove that f is a C^∞ function for all $x \in \mathbb{R}$. (*b*) Prove that $f(x)$ is not analytic by showing that its Taylor series at $x_0 = 0$ does not converge to $f(x)$ when $x \neq 0$.

11.3.22. Consider the ordinary differential equation $2x\,u'' + u' + x\,u = 0$. (a) Prove that $x = 0$ is a regular singular point. (*b*) Find two independent series solutions in powers of x.

11.3.23. Consider the differential equation $\dfrac{u''}{2-x} = \dfrac{u}{x^2}$. (a) Classify all $x_0 \in \mathbb{R}$ as either a (*i*) regular point; (*ii*) regular singular point; and/or (*iii*) irregular singular point. Explain your answers. (*b*) Find a series solution to the equation based at the point $x_0 = 0$, or explain why none exists. What is the radius of convergence of your series?

11.3.24. Consider the differential equation $u'' + \left(1 - \dfrac{1}{x}\right) u' + u = 0$.

(a) Classify all $x_0 \in \mathbb{R}$ as either (*i*) a regular point; (*ii*) a regular singular point; (*iii*) an irregular singular point; (*iv*) none of the above. Explain your answers.

(*b*) Write out the first five nonzero terms in a series solution.

11.3.25. Consider the differential equation $4x\,u'' + 2u' + u = 0$. (a) Classify the values of x for which the equation has regular points, regular singular points, and irregular singular points. (*b*) Find two independent series solutions, in powers of x. For what values of x do your series converge? (*c*) By inspection of your series, write the general solution to the equation in terms of elementary functions.

11.3.26. The *Chebyshev differential equation* is $(1-x^2)u'' - x\,u' + m^2 u = 0$. ($a$) Find all ($i$) regular points; ($ii$) regular singular points; (iii) irregular singular points. (b) Show that if m is an integer, the equation has a polynomial solution of degree m, known as a *Chebyshev polynomial*. Write down the Chebyshev polynomials of degrees $1, 2$, and 3. (c) For $m = 1$, find two linearly independent series solutions based at the point $x_0 = 1$.

11.3.27. Write the following Bessel functions in terms of elementary functions:
(a) $J_{5/2}(x)$, (b) $J_{7/2}(x)$, (c) $J_{-3/2}(x)$.

11.3.28. Prove the identity (11.103).

11.3.29. Suppose that $u(x)$ solves Bessel's equation. (a) Find a second order ordinary differential equation satisfied by the function $w(x) = \sqrt{x}\,u(x)$. (b) Use this result to rederive the formulas for $J_{1/2}(x)$ and $J_{-1/2}(x)$.

11.3.30. Let $m \geq 0$ be real, and consider the *modified Bessel equation* of order m:

$$x^2\,u'' + x\,u' - (x^2 + m^2)\,u = 0. \tag{11.114}$$

(a) Explain why $x_0 = 0$ is a regular singular point.
(b) Use the method of Frobenius to construct a series solution based at $x_0 = 0$. Can you relate your solutions to the Bessel function $J_m(x)$?

11.3.31. (a) Let a, b, c be constants with $b, c \neq 0$. Show that the function $u(x) = x^a J_0(b\,x^c)$ solves the ordinary differential equation

$$x^2\,\frac{d^2u}{dx^2} + (1-2a)\,x\,\frac{du}{dx} + (b^2c^2x^{2c} + a^2)\,u = 0.$$

What is the general solution to this equation?
(b) Find the general solution to the ordinary differential equation

$$x^2\,\frac{d^2u}{dx^2} + \alpha\,x\,\frac{du}{dx} + (\beta\,x^{2c} + \gamma)\,u = 0,$$

for constants α, β, γ, c with $\beta, c \neq 0$.

11.3.32. Let $k > 0$ be a constant. The ordinary differential equation $\dfrac{d^2u}{dt^2} + e^{-2t}\,u = 0$ describes the vibrations of a weakening spring whose stiffness $k(t) = e^{-2t}$ is exponentially decaying in time. (a) Show that this equation can be solved in terms of Bessel functions of order 0. *Hint*: Perform a change of variables. (b) Does the solution tend to 0 as $t \to \infty$?

11.3.33. We know that $\widehat{u}(x) = J_0(x)$ is a solution to the Bessel equation of order 0, namely

$$x u'' + u' + x u = 0. \tag{11.115}$$

In accordance with the general Frobenius method, construct a second solution of the form

$$\widetilde{u}(x) = J_0(x) \log x + \sum_{n=1}^{\infty} v_n x^n .$$

11.3.34. Is it possible to have all solutions to an ordinary differential equation bounded at a regular singular point? If not, explain why not. If true, give an example where this happens.

11.4 The Heat Equation in a Disk

11.4.1. At the initial time $t_0 = 0$, a concentrated unit heat source is instantaneously applied at position $x = \frac{1}{2}$, $y = 0$, to a circular metal disk of unit radius and unit thermal diffusivity whose outside edge is held at 0°. Write down an eigenfunction series for the resulting temperature distribution at time $t > 0$. *Hint*: Be careful working with the delta function in polar coordinates; see Exercise 6.3.6.

11.4.2. Solve Exercise 11.4.1 when the concentrated unit heat source is instantaneously applied at the center of the disk.

11.4.3. (a) Write down the Fourier–Bessel series for the solution to the heat equation on a unit disk with $\gamma = 1$, whose circular edge is held at 0° and subject to the initial conditions $u(0, x, y) \equiv 1$ for $x^2 + y^2 \le 1$. *Hint*: Use (11.112) to evaluate the integrals for the coefficients. (b) Approximate the time $t_\star \ge 0$ after which the temperature of the disk is everywhere $\le .5^\circ$.

11.4.4. (a) Write down the first three nonzero terms in the Fourier–Bessel series for the solution to the heat equation on a unit disk with $\gamma = 1$ whose circular edge is held at 0° subject to the initial conditions $u(0, r, \theta) = 1 - r$ for $r \le 1$. Use numerical integration to evaluate the coefficients. (b) Use your approximation to determine at which times $t \ge 0$ the temperature of the disk is everywhere $\le .5^\circ$.

11.4.5. Prove that every separable eigenfunction of the Dirichlet boundary value problem for the Helmholtz equation in the unit disk can be written in the form

$$c\, J_m(\zeta_{m,n} r) \cos(m\theta - \alpha) \quad \text{for fixed } c \ne 0 \text{ and } -\pi < \alpha \le \pi.$$

11.4.6. Suppose the initial data $f(r, \theta)$ in (11.49) satisfies $\int_0^1 \int_{-\pi}^{\pi} f(r, \theta)\, J_0(\zeta_{0,1} r)\, r\, d\theta\, dr = 0$.
(a) What is the decay rate to equilibrium of the resulting heat equation solution $u(t, r, \theta)$?
(b) Prove that, generically, the asymptotic temperature distribution has half the disk above the equilibrium temperature and the other half below. Can you predict the diameter that separates the two halves? (c) If you know that $a_{0,1} = 0$, and also that the long-time temperature distribution is radially symmetric, what is the (generic) decay rate? What is the asymptotic temperature distribution?

11.4.7. Show how to use a scaling symmetry to solve the heat equation in a disk of radius R knowing the solution in a disk of radius 1.

11.4.8. Use rescaling, as in Exercise 11.4.7, to produce the solution to the Dirichlet initial-boundary value problem for a disk of radius 2 with diffusion coefficient $\gamma = 5$.

11.4.9. If it takes a disk of unit radius 3 minutes to reach (approximate) thermal equilibrium, how long will it take a disk of radius 2 made out of the same material and subject to the same homogeneous boundary conditions to reach equilibrium?

11.4.10. Assuming Dirichlet boundary conditions, does a square or a circular disk of the same area reach thermal equilibrium faster? Use your intuition first, and then check using the explicit formulas.

11.4.11. Answer Exercise 11.4.10 when the square and circle have the same perimeter.

11.4.12. Which reaches thermal equilibrium faster: a disk whose edge is held at 0° or a disk of the same radius that is fully insulated?

11.4.13. A circular metal disk is removed from an oven and then fully insulated. *True or false*: (*a*) The eventual equilibrium temperature is constant. (*b*) For large $t \gg 0$, the temperature $u(t, x, y)$ becomes more and more radially symmetric. If false, what can you say about the temperature profile at large times?

11.4.14. (*a*) Write down an eigenfunction series formula for the temperature dynamics of a disk of radius 1 that has an insulated boundary. (*b*) What is the eventual equilibrium temperature? (*c*) Is the rate of decay to thermal equilibrium (*i*) faster, (*ii*) slower, or (*iii*) the same as a disk with Dirichlet boundary conditions?

11.4.15. Write out a series solution for the temperature in a half-disk of radius 1, subject to (*a*) homogeneous Dirichlet boundary conditions on its entire boundary; (*b*) homogeneous Dirichlet conditions on the circular part of its boundary and homogeneous Neumann conditions on the straight part. (*c*) Which of the two boundary conditions results in a faster return to equilibrium temperature? How much faster?

11.4.16. A large sheet of metal is heated to 100°. A circular disk and a semi-circular half-disk of the same radius are cut out of it. Their edges are then held at 0°, while being fully insulated from above and below.

(*a*) *True or false*: The half-disk goes to thermal equilibrium twice as fast as the disk.

(*b*) If you need to wait 20 minutes for the circular disk to cool down enough to be picked up in your bare hands, how long do you need to wait to pick up the semi-circular disk?

11.4.17. Two identical plates have the shape of an annular ring $\{1 < r < 2\}$ with inner radius 1 and outer radius 2. The first has an insulated inner boundary and outer boundary held at 0°, while the second has an insulated outer boundary and inner boundary held at 0°. If both start out at the same temperature, which reaches thermal equilibrium faster? Quantify the rates of decay.

11.4.18. Let $m \geq 0$ be a nonnegative integer. In this exercise, we investigate the completeness of the eigenfunctions of the Bessel boundary value problem (11.56–57). To this end, define the Sturm–Liouville linear differential operator

$$S[u] = -\frac{1}{x}\frac{d}{dx}\left(x\,\frac{du}{dx}\right) + \frac{m^2}{x^2}\,u,$$

subject to the boundary conditions $|\,u'(0)\,| < \infty$, $u(1) = 0$, and either $|\,u(0)\,| < \infty$ when $m = 0$, or $u(0) = 0$ when $m > 0$.

(a) Show that S is self-adjoint relative to the inner product $\langle\, f\,, g\,\rangle = \int_0^1 f(x)\,g(x)\,x\,dx$.

(b) Prove that the eigenfunctions of S are the rescaled Bessel functions $J_m(\zeta_{m,n}\,x)$ for $n = 1, 2, 3, \ldots$. What are the orthogonality relations?

(c) Find the Green's function $G(x;\xi)$ and modified Green's function $\widehat{G}(x;\xi)$, cf. (9.59), associated with the boundary value problem $S[\,u\,] = 0$.

(d) Use the criterion of Theorem 9.47 to prove that the eigenfunctions are complete.

11.4.19. Determine the Bessel roots $\zeta_{1/2,n}$. Do they satisfy the asymptotic formula (11.119)?

11.4.20. Use a numerical root finder to compute the first 10 Bessel roots $\zeta_{3/2,n}$, $n = 1, \ldots, 10$. Compare your values with the asymptotic formula (11.119).

11.4.21. Prove that $J_{m-1}(\zeta_{m,n}) = -\,J_{m+1}(\zeta_{m,n})$.

11.4.22. In this exercise, we prove formula (11.126).

(a) First, use the recurrence formulae (11.111) to prove

$$\frac{d}{dx}\left[\,x^2\left(\,J_m(x)^2 - J_{m-1}(x)\,J_{m+1}(x)\,\right)\right] = 2x\,J_m(x)^2.$$

(b) Integrate both sides of the previous formula from 0 to the Bessel zero $\zeta_{m,n}$ and then use Exercise 11.4.21 to show that

$$\int_0^{\zeta_{m,n}} x\,J_m(x)^2\,dx = -\,\frac{\zeta_{m,n}^2}{2}\,J_{m-1}(\zeta_{m,n})\,J_{m+1}(\zeta_{m,n}) = \frac{\zeta_{m,n}^2}{2}\,J_{m+1}(\zeta_{m,n})^2.$$

(c) Next, use a change of variables to establish the identity

$$\int_0^1 z\,J_m(\zeta_{m,n}\,z)^2\,dz = \tfrac{1}{2}\,J_{m+1}(\zeta_{m,n})^2.$$

(d) Finally, use the formulae for $v_{m,n}$ and $\widehat{v}_{m,n}$ to complete the proof of (11.126).

11.4.23. Prove directly that the eigenfunctions $v_{m,n}(r,\theta)$ and $\widehat{v}_{m,n}(r,\theta)$ in (11.122) are orthogonal with respect to the L^2 inner product on the unit disk.

11.4.24. Establish the following alternative formulae for the eigenfunction norms:

$$\| v_{0,n} \| = \sqrt{\pi}\, \left| J_0'(\zeta_{0,n}) \right|, \qquad \| v_{m,n} \| = \| \widehat{v}_{m,n} \| = \sqrt{\frac{\pi}{2}}\, \left| J_m'(\zeta_{m,n}) \right|.$$

11.5 The Fundamental Solution to the Planar Heat Equation

11.5.1. Solve the following initial value problem: $u_t = 5\,(u_{xx} + u_{yy}), \quad u(0,x,y) = e^{-(x^2+y^2)}$.

11.5.2. Write down an integral formula for the solution to the following initial value problem:

$$u_t = 3(u_{xx} + u_{yy}), \qquad u(0,x,y) = (1 + x^2 + y^2)^{-2}.$$

11.5.3. At the initial time $t = 0$, a unit heat source is instantaneously applied at the origin of the (x,y)–plane. For $t > 0$, what is the maximum temperature experienced at a point $(x,y) \neq \mathbf{0}$? At what time is the maximum temperature achieved? Does the temperature approach an equilibrium value as $t \to \infty$? If so, how fast?

11.5.4. (a) Find an eigenfunction series representation of the fundamental solution for the heat equation $u_t = \Delta u$ on the unit square $\{0 \le x, y \le 1\}$ when subject to homogeneous Dirichlet boundary conditions. (b) Write the solution to the initial value problem $u(0,x,y) = f(x,y)$ in terms of the fundamental solution. (c) Discuss how your formula is related to the Fourier series solution (11.43).

11.5.5. Let $u(t,x,y)$ be a solution to the heat equation on all of $\mathbb{R}^2$ such that u and $\| \nabla u \| \to 0$ rapidly as $\| \mathbf{x} \| \to \infty$. (a) Prove that the total heat $H(t) = \iint u(t,x,y)\, dx\, dy$ is constant. (b) Explain how this can be reconciled with the statement that $u(t,x,y) \to 0$ as $t \to \infty$ at all points $(x,y) \in \mathbb{R}^2$.

11.5.6. Consider the initial value problem $u_t = \gamma\,\Delta u + H(t,x,y), \ u(0,x,y) = 0$, for the inhomogeneous heat equation on the entire (x,y)–plane, where $H(t,x,y)$ represents a time-varying external heat source. Derive an integral formula for its solution. *Hint*: Mimic the solution method in Section 8.1.

11.5.7. A flat plate of infinite extent with unit thermal diffusivity starts off at 0°. From then on, a unit heat source is continually applied at the origin. Find the resulting temperature distribution. Does the temperature eventually reach a steady state? *Hint*: Use Exercise 11.5.6.

11.5.8. Building on Example 11.14, we model the "diffusion" of a set $D \subset \mathbb{R}^2$ as the solution $u(t,x,y)$ to the heat equation $u_t = \Delta u$ subject to the initial condition $u(0,x,y) = \chi_D(x,y)$, where $\chi_D(x,y) = \begin{cases} 1, & (x,y) \in D, \\ 0, & (x,y) \notin D, \end{cases}$ is the *characteristic function* of the set D.
(*a*) Write down a formula for the diffusion of the set D.
(*b*) *True or false*: At each t, the diffusion $u(t,x,y)$ is the characteristic function of a set D_t.
(*c*) Prove that $0 < u(t,x,y) < 1$ for all (x,y) and $t > 0$. (*d*) What is $\lim_{t \to \infty} u(t,x,y)$?
(*e*) Write down a formula for the diffusion of a unit square $D = \{0 \le x, y \le 1\}$, and then plot the result at several times. Discuss what you observe.

11.5.9. (*a*) Explain why the delta function on $\mathbb{R}^2$ satisfies the scaling law $\delta(x,y) = \beta^2\,\delta(\beta x, \beta\, y)$ for $\beta > 0$. (*b*) Verify that the fundamental solution to the heat equation on $\mathbb{R}^2$ obeys the same scaling law: $F(t,x,y) = \beta^2\, F(\beta^2 t, \beta\, x, \beta\, y)$. (*c*) Is the fundamental solution a similarity solution?

11.5.10. (*a*) Find the fundamental solution on $\mathbb{R}^2$ to the cable equation $u_t = \gamma\,\Delta u - \alpha\, u$, where $\alpha > 0$ is constant. (*b*) Use your solution to write down a formula for the solution to the general initial value problem $u(0,x,y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$.

11.5.11. (*a*) Prove that if $v(t,x)$ and $w(t,x)$ solve the dispersive wave equation (8.90), then their product $u(t,x,y) = v(t,x)\, w(t,y)$ solves the two-dimensional dispersive equation

$$u_t + u_{xxx} + u_{yyy} = 0.$$

(*b*) What is the fundamental solution on $\mathbb{R}^2$ of the latter equation? (*c*) Write down an integral formula for the solution to the initial value problem $u(0,x,y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$.

11.5.12. Define the two-dimensional convolution $f * g$ of functions $f(x,y)$ and $g(x,y)$ so that equation (11.135) is valid.

11.6 The Planar Wave Equation

11.6.1. Use your physical intuition to decide whether the following statements are *true or false*. Then justify your answer.
(*a*) Increasing the stiffness of a membrane increases the wave speed.
(*b*) Increasing the density of a membrane increases the wave speed.
(*c*) Increasing the size of a membrane increases the wave speed

11.6.2. Two uniform membranes have the same shape, but are made out of different materials. Assuming that they are both subject to the same homogeneous boundary conditions, how are their vibrational frequencies related?

11.6.3. List the numerical values of the six lowest vibrational frequencies of a unit square with wave speed $c = 1$ when subject to homogeneous Dirichlet boundary conditions. How many linearly independent normal modes are associated with each of these frequencies?

11.6.4. The rectangular membrane $R = \{-1 < x < 1,\ 0 < y < 1\}$ has its two short sides attached to the (x,y)–plane, while its long sides are left free. The membrane is initially displaced so that its right half is one unit above, while its left half is one unit below the plane, and then released with zero initial velocity. (This discontinuous initial data serves to model a very sharp transition region.) Assume that the physical units are chosen so the wave speed $c = 1$. (a) Write down an initial-boundary value problem that governs the vibrations of the membrane. (b) What are the fundamental frequencies of vibration of the membrane? (c) Find the eigenfunction series solution that describes the subsequent motion of the membrane. (d) Is the motion (i) periodic? (ii) quasiperiodic? (iii) unstable? (iv) chaotic? Explain your answer.

11.6.5. Determine the solution to the following initial-boundary value problems for the wave equation on the rectangle $R = \{0 < x < 2,\ 0 < y < 1\}$:

(a) $$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0, \\ u(0,x,y) = \sin \pi y, & u_t(0,x,y) = \sin \pi y; \end{cases}$$

(b) $$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = \dfrac{\partial u}{\partial x}(t,0,y) = \dfrac{\partial u}{\partial x}(t,2,y) = 0, \\ u(0,x,y) = \sin \pi y, & u_t(0,x,y) = \sin \pi y; \end{cases}$$

(c) $$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0, \\ u(0,x,y) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2, \end{cases} & u_t(0,x,y) = 0; \end{cases}$$

(d) $$\begin{cases} u_{tt} = 2u_{xx} + 2u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0, \\ u(0,x,y) = 0, & u_t(0,x,y) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2. \end{cases} \end{cases}$$

11.6.6. *True or false*: The more sides of a rectangle that are tied down, the faster it vibrates.

11.6.7. Answer Exercise 11.6.3 when (a) two adjacent sides of the square are tied down and the other two are left free; (b) two opposite sides of the square are tied down and the other two are left free; (c) the membrane is freely floating in outer space.

11.6.8. A square drum has two sides fixed to a support and two sides left free. Does the drum vibrate faster if the fixed and free sides are adjacent to each other or on opposite sides?

11.6.9. Write down a periodic solution to the wave equation on a unit square, subject to homogeneous Dirichlet boundary conditions, that is *not* a normal mode. Does it vibrate at a fundamental frequency?

11.6.10. A rectangular drum with side lengths 1 cm by 2 cm and unit wave speed $c = 1$ has its boundary fixed to the (x,y)–plane while subject to a periodic external forcing of the form $F(t,x,y) = \cos(\omega t)\, h(x,y)$. (a) At which frequencies ω will the forcing incite resonance in the drum? (b) If ω is a resonant frequency, write down the condition(s) on $h(x,y)$ that ensure excitation of a resonant mode.

11.6.11. The right half of a rectangle of side lengths 1 by 2 is initially displaced, while the left half is quiescent. *True or false*: The ensuing vibrations are restricted to the right half of the membrane.

11.6.12. A torus (inner tube) can be obtained by gluing together each of the two pairs of opposite sides of a rubber rectangle. The (small) vibrations of the torus are described by the following periodic initial-boundary value problem for the wave equation, in which x, y represent angular variables:

$$\begin{array}{lll} u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}), & u(0,x,y) = f(x,y), & u_t(0,x,y) = g(x,y), \\ u(t,-\pi,y) = u(t,\pi,y), & u_x(t,-\pi,y) = u_x(t,\pi,y), & -\pi < x < \pi, \\ u(t,x,-\pi) = u(t,x,\pi), & u_x(t,x,-\pi) = u_x(t,x,\pi), & -\pi < y < \pi. \end{array}$$

(*a*) Find the fundamental frequencies and normal modes of vibration. (*b*) Write down a series for the solution. (*c*) Discuss the stability of a vibrating torus. Is the motion (*i*) periodic; (*ii*) quasiperiodic; (*iii*) chaotic; (*iv*) none of these?

11.6.13. The *forced wave equation* $u_{tt} = c^2 \Delta u + F(x,y)$ on a bounded domain $\Omega \subset \mathbb{R}^2$ models a membrane subject to a constant external forcing function $F(x,y)$. Write down an eigenfunction series solution to the forced wave equation when the membrane is subject to homogeneous Dirichlet boundary conditions and initial conditions $u(0,x,y) = f(x,y)$, $u_t(0,x,y) = g(x,y)$. *Hint*: Expand the forcing function in an eigenfunction series.

11.6.14. A circular drum of radius $\zeta_{0,1} \approx 2.4048$ has initial displacement and velocity

$$u(0,x,y) = 0, \qquad \frac{\partial u}{\partial t}(0,x,y) = 2\, J_0\left(\sqrt{x^2+y^2}\,\right).$$

Assuming that the circular edge of the drum is fixed to the (x,y)–plane, describe, both qualitatively and quantitatively, its subsequent motion.

11.6.15. Write out the integral formulae for the coefficients in the Fourier–Bessel series solution (11.159) to the wave equation in a circular disk in terms of the initial data $u(0,r,\theta) = f(r,\theta)$, $u_t(0,r,\theta) = g(r,\theta)$.

11.6.16. A circular drum at rest is struck with a concentrated blow at its center. Write down an eigenfunction series describing the resulting vibration.

11.6.17. (a) Set up and solve the initial-boundary value problem for the vibrations of a uniform circular drum of unit radius that is freely floating in space. (*b*) Discuss the stability of the drum's motion. (*c*) Are the vibrations slower or faster than when its edges are fixed to a plane?

11.6.18. A flat quarter-disk of radius 1 has its circular edge and one of its straight edges attached to the (x,y)–plane, while the other straight edge is left free. At time $t = 0$ the disk is struck with a hammer (unit delta function) at its midpoint, i.e., at radius $\frac{1}{2}$ and halfway between the straight edges. (a) Write down an initial-boundary value problem for the subsequent vibrations of the quarter-disk. *Hint*: Be careful with the form of the delta function in polar coordinates; see Exercise 6.3.6. (*b*) Assuming that the physical units are chosen so that the wave speed $c = 1$, determine the quarter-disk's vibrational frequencies. (*c*) Write down an eigenfunction series solution for the subsequent motion. (*d*) Is the motion unstable? periodic? If so, what is the period?

11.6.19. *True or false*: Assuming homogeneous Dirichlet boundary conditions, the fundamental frequencies of a vibrating half-disk are exactly twice those of the full disk of the same radius.

11.6.20. The edge of a circular drum is moved periodically up and down, so $u(t,1,\theta) = \cos\omega t$. Assuming that the drum is initially at rest, discuss its response.

11.6.21. A drum is in the shape of a circular annulus with outer radius 1 meter and inner radius .5 meter. Find numerical values for its first three fundamental vibrational frequencies.

11.6.22. A homogeneous rope of length 1 and weight 1 is suspended from the ceiling. Taking x as the vertical coordinate, with $x = 1$ representing the fixed end and $x = 0$ the free end, the planar displacement $u(t,x)$ of the rope satisfies the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(x\,\frac{\partial u}{\partial x}\right), \qquad \begin{array}{ll} |\,u(t,0)\,| < \infty, & u(t,1) = 0, \\ u(0,x) = f(x), & \dfrac{\partial u}{\partial t}(0,x) = g(x), \end{array} \qquad t > 0, \quad 0 < x < 1.$$

(*a*) Find the solution. *Hint*: Let $y = \sqrt{x}$. (*b*) Are the vibrations periodic or quasiperiodic? (*c*) Describe the behavior of the rope when subject to uniform periodic external forcing

$$F(t,x) = a\cos\omega t.$$

11.6.23. *True or false*: Two rectangular membranes, made out of the same material and both subject to Dirichlet boundary conditions, have the same relative vibrational frequencies if and only if they are have similar shapes.

11.6.24. *True or false*: (*a*) The vibrational frequencies of a square with side lengths $a = b = 2$ are four times as slow as those of a square with side lengths $a = b = 1$.
(*b*) The vibrational frequencies of a rectangle with side lengths $a = 2$, $b = 1$, are twice as slow as those of a square with side lengths $a = b = 1$.

11.6.25. A vibrating rectangle of unknown size has wave speed $c = 1$ and is subject to homogeneous Dirichlet boundary conditions. How many of its lowest vibrational frequencies do you need to know in order to determine the size of the rectangle?

11.6.26. Answer Exercise 11.6.25 when the rectangle is subject to homogeneous Neumann boundary conditions.

11.6.27. A circular drum has the A above middle C, which has a frequency of 440 Hertz, as its lowest tone. What notes are the first five overtones nearest? Try playing these on a piano or guitar. Or, if you have a synthesizer, try assembling notes of these frequencies to see how closely it reproduces the dissonant sound of a drum.

11.6.28. In an orchestra, a circular snare drum of radius 1 foot sits near a second circular drum made out of the same material. Vibrations of the first drum are observed to excite an undesired resonant vibration in its partner. What are the possible radii of the second drum?

11.6.29. *True or false*: The relative vibrational frequencies of a half-disk, subject to Dirichlet boundary conditions, are a subset of the relative vibrational frequencies of a full disk.

11.6.30. *True or false*: If $u(t,x,y) = \cos(\omega t)\, v(x,y)$ is a normal mode of vibration for a unit square subject to homogeneous Dirichlet boundary conditions, then the function $\widehat{u}(t,x,y) = \cos(\omega t)\, v\left(\frac{1}{2}x, \frac{1}{3}y\right)$ is a normal mode of vibration for a 2×3 rectangle that is subject to the same boundary conditions, but with a possibly different wave speed. If true, how are the wave speeds of the two rectangles related?

11.6.31. Prove that if $u(t,x,y)$ is a solution to the two-dimensional wave equation, so is the translated function $U(t,x,y) = u(t - t_0, x - x_0, y - y_0)$, for any constants t_0, x_0, y_0.

11.6.32. (*a*) Prove that if $u(t,x,y)$ solves the wave equation, so does $U(t,x,y) = u(-t,x,y)$. Thus, unlike the heat equation, the wave equation is time-reversible, and its solutions can be unambiguously followed backwards in time. (*b*) Suppose $u(t,x,y)$ solves the initial value problem (11.141). Write down the initial value problem satisfied by $U(t,x,y)$.

11.6.33. (*a*) Prove that, on $\mathbb{R}^2$, the solution to the pure displacement initial value problem $u_{tt} = c^2 \Delta u$, $u(0,x,y) = f(x,y)$, $u_t(0,x,y) = 0$, is an even function of t.
(*b*) Prove that the solution to the pure velocity initial value problem $u_{tt} = c^2 \Delta u$, $u(0,x,y) = 0$, $u_t(0,x,y) = g(x,y)$, is an odd function of t. *Hint*: Use Exercise 11.6.32 and uniqueness of solutions to the initial value problem.

11.6.34. Suppose $v(t,x)$ is any solution to the one-dimensional wave equation $v_{tt} = v_{xx}$. Prove that $u(t,x,y) = v(t, a\,x + b\,y)$, for any constants $(a,b) \neq (0,0)$, solves the two-dimensional wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ for some choice of wave speed. Describe the behavior of such solutions.

11.6.35. A *traveling-wave solution* to the two-dimensional wave equation has the form $u(t,x,y) = v(x - a\,t, y - a\,t)$, where a is a constant. Find the partial differential equation satisfied by the function $v(\xi, \eta)$. Is the equation hyperbolic?

11.6.36. Is the counterpart of Lemma 11.11 valid for the wave equation? In other words, if $v(t,x)$ and $w(t,x)$ are any two solutions to the one-dimensional wave equation, is their product $u(t,x,y) = v(t,x)\, w(t,y)$ a solution to the two-dimensional wave equation?

11.6.37. (*a*) How would you solve an initial-boundary value problem for the wave equation on a rectangle that is not aligned with the coordinate axes? (*b*) Apply your method to set up and solve an initial-boundary value problem on the square $R = \{\,|\,x + y\,| < 1,\ |\,x - y\,| < 1\,\}$.

Chladni Figures and Nodal Curves

♢ 11.6.38. Suppose that a membrane is vibrating in a normal mode. Prove that the membrane lies instantaneously completely flat at regular time intervals.

♢ 11.6.39. For a vibrating disk of unit radius, determine the radius of the circular nodal curve for the next-to-lowest circular mode.

11.6.40. Order the five nodal circles displayed in Figure 11.12 according to their size.

11.6.41. Sketch the Chladni figures in a unit disk corresponding to the following vibrational frequencies. Determine numerical values for the radii of any circular nodal curves.
(a) $\omega_{4,0}$, (*b*) $\omega_{4,2}$, (*c*) $\omega_{2,4}$, (*d*) $\omega_{3,3}$, (*e*) $\omega_{1,5}$.

11.6.42. *True or false*: Any diameter of a circular disk is a nodal curve for some normal mode.

11.6.43. *True or false*: The nodal curves on a semicircular disk are all semicircles and rays emanating from the center.

11.6.44. (a) Find the smallest distinct pair of positive integers $(k,l) \neq (m,n)$ satisfying (11.172) that are not obtained by simply reversing the order, i.e., $(k,l) \neq (n,m)$. (*b*) Find the next-smallest example. (*c*) Plot two or three Chladni figures arising from such degenerate eigenfunctions.

11.6.45. Let R be a rectangle all of whose sides are fixed to the (x,y)–plane. Suppose that all its nodal curves are straight lines. What can you say about its side lengths a,b?

11.6.46. *True or false*: The nodal regions of a vibrating rectangle are similarly shaped rectangles.

11.6.47. Prove that any point of intersection (x_0,y_0) of two nodal curves associated with the same normal mode is a critical point of the associated eigenfunction: $\nabla v(x_0,y_0) = \mathbf{0}$.

11.6.48. *True or false*: The nodal curves on a domain do not depend on the choice of boundary conditions.