§ Separation of Variables 變數分離法(Fourier series)

$u_{tt}=c^2u_{xx},0<x<l,t>0$ 波動方程

$u(0,t)=u(l,t)=0$ ， $t\ge 0$
$u(x,0)=f(x),u_t(x,0)=g(x),0\le x\le l$

設解為 $u(x,t)=X(x)T(t)$ 則 $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{c^2T(t)}=\lambda$

若 $\lambda>0$ 則 $X(x)=Ae^{\sqrt{\lambda}x}+Be^{-\sqrt{\lambda}x}$ ， $T(t)=Ce^{c\sqrt{\lambda}t}+De^{-c\sqrt{\lambda}t}$

無法描述週期現象 $\lambda=0$ 亦同。
若 $\lambda<0$ 令 $\lambda=-\omega^2$ 則 $X(x)=A\cos\omega x+B\sin\omega x$ ， $T(t)=C\cos c\omega t+D\sin c\omega t$
$u(x,t)=X(x)T(t)=(A\cos\omega x+B\sin\omega x)(C\cos c\omega t+D\sin c\omega t)$

可描述頻率為 $\dfrac{c\omega}{2\pi}$ 的振盪運動。
邊界條件及初始條件
$u(0,t)=X(0)T(t)=0,u(l,t)=X(l)T(t)=0,t\ge 0$
因為 $T(t)\ne 0$ (否則僅能得到一個滿足上述邊界條件的零函數解 $u(x,t)=0$，不合。)
故 $X(0)=A=0,X(l)=B\sin\omega l=0$

又， $B\ne 0$ (否則 y(x,t)僅能為零函數)，所以 $\sin\omega l=0$ 或 $\omega=\dfrac{n\pi}{l},n=1,2,3,...$

因此，對應每一個 n，可以得到 (10-5-1) 式在上述所予邊界條件下的特解，

$$u_n(x,\ t)=\left(C_n\cos\frac{nc\pi}{l}t+D_n\sin\frac{nc\pi}{l}t\right)\sin\frac{n\pi x}{l}$$

滿足所予初始條件的特別解為無窮級數
$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}(C_n\cos\frac{nc\pi}{l}t+\mathrm{D}_n\sin\frac{nc\pi}{l}t)\sin\frac{n\pi x}{l}$$
假設此式收斂，則由所於初始條件知
$$u(x,0)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}=f(x)$$
$$u_t(x,0)=\sum_{n=1}^{\infty}\frac{nc\pi}{l}D_n\sin\frac{n\pi x}{l}=g(x),0\le x\le l$$

可知 C_n 與 $\dfrac{nc\pi}{l}D_n$ 分別為 f(x)與 g(x)的半幅正弦展開式的係數，即

$$C_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,n=1,2,3,...$$

$$D_n = \frac{2}{nc\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}dx, n = 1, 2, 3, \ldots$$

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這裡只處理兩變數的方程式。

1. Wave equations：sound waves，water waves，elastic waves，electromagnetic waves and so on。(hyperbolic class)
2. Heat equation models diffusion processes，including thermal energy in solids，solutes in liquids，and biological populations。(parabolic class)
3. Laplace equation and its inhomogeneous counterpart，the Poisson equation，both variables represent space coordinates ，x and y，and the associated boundary value problems model the equilibrium configuration of a planar body，e.g. the deformations of a membrane。(elliptic class)

   The solutions of Laplace equation are known as harmonic functions。

4.1 The Diffusion and Heat Equations

c.f. PDE103HeatEquation1-2

The separation solutions to the heat equation are based on the exponential ansatz

$$u(t,x) = e^{-\lambda t}v(x) \Rightarrow -\gamma\frac{d^2v}{dx^2} = \lambda v$$

Each nontrivial solution $v(x) \neq 0$ is an eigenfunction，with associated eigenvalue λ，for the linear differential operator $L(v) = -\gamma v''(x)$

$$\begin{cases} u_t - ku_{xx} = f(x,t), t > 0 \\ u\big|_{t=0} = g(x), 0 \leq x \leq l \\ u(t,0) = u(t,l) = 0, t > 0 \end{cases}$$

The eigenfunction are founded by solning the Dirichlet boundary value probles $\gamma\frac{d^2v}{dx^2} + \lambda v = 0, v(0) = v(l) = 0$

If λ is either complex，or real and nonpositive，then the only solution to the boundary value problem is the trivial solution $v(x) \equiv 0$。This means that all the eigenvalues need necessarily be real and positive。

When $\lambda > 0$，the general solution is a trigonometric function $v(t) = a\cos\omega t + b\sin\omega t$，where $\omega = \sqrt{\frac{\lambda}{\gamma}}$

and a and b are arbitrary constants. The first boundary condition requires $v(0) = a = 0$. This serves to eliminate the cosine term, and then the second boundary condition requires

$$v(\ell) = b \sin \omega \ell = 0.$$

Therefore, since we require $b \neq 0$ — otherwise, the solution is trivial and does not qualify as an eigenfunction — $\omega \ell$ must be an integer multiple of π, and so

$$\omega = \frac{\pi}{\ell}, \qquad \frac{2\pi}{\ell}, \qquad \frac{3\pi}{\ell}, \qquad \ldots .$$

We conclude that the eigenvalues and eigenfunctions of the boundary value problem (4.20) are

$$\lambda_n = \gamma \left(\frac{n\pi}{\ell} \right)^2, \qquad v_n(x) = \sin \frac{n\pi x}{\ell}, \qquad n = 1, 2, 3, \ldots . \tag{4.21}$$

The corresponding eigensolutions (4.18) are

$$u_n(t,x) = \exp\left(- \frac{\gamma n^2 \pi^2 t}{\ell^2} \right) \sin \frac{n\pi x}{\ell}, \qquad n = 1, 2, 3, \ldots . \tag{4.22}$$

Each represents a trigonometrically oscillating temperature profile that maintains its form while decaying to zero at an exponentially fast rate.

To solve the general initial value problem, we assemble the eigensolutions into an infinite series,

$$u(t,x) = \sum_{n=1}^{\infty} b_n \, u_n(t,x) = \sum_{n=1}^{\infty} b_n \exp\left(- \frac{\gamma n^2 \pi^2 t}{\ell^2} \right) \sin \frac{n\pi x}{\ell}, \tag{4.23}$$

whose coefficients b_n are to be fixed by the initial conditions. Indeed, assuming that the series converges, the initial temperature profile is

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} = f(x). \tag{4.24}$$

This has the form of a Fourier sine series (3.52) on the interval $[0, \ell]$. Thus, the coefficients are determined by the Fourier formulae (3.53), and so

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx, \qquad n = 1, 2, 3, \ldots . \tag{4.25}$$

The resulting formula (4.23) describes the Fourier sine series for the temperature $u(t,x)$ of the bar at each later time $t \geq 0$.

§ Smoothing and long-time behavior

§ The heated ring redux

§ inhomogeneous boundary conditions

§ Robin boundary conditions

§ The root cellar problem