

§ 二階 常係數 PDE

齊次 $au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \dots (*)$

1. Hyperbolic type $b^2 - 4ac > 0$ 例 波動方程式 $c^2u_{xx} - u_{yy} = 0$
2. Parabolic type $b^2 - 4ac = 0$ 例 擴散方程式 $ku_{xx} - u_y = 0, k > 0$
3. Elliptic type $b^2 - 4ac < 0$ 例 Laplace 方程 $u_{xx} + u_{yy} = 0$

Let $u = Ae^{\alpha x + \beta y}$ 代入(*) 則 $a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f = 0$

例

1. 解 $2u_x + 3u_y - 2u = 0$

設 $u = Ae^{\alpha x + \beta y}$ 則 $2\alpha + 3\beta - 2 = 0$

$$e^{\alpha x + \beta y} = e^{\left(\frac{2-3\beta}{2}\right)x + \beta y} = e^x \cdot e^{\frac{\beta}{2}(2y-3x)}$$

所以 通解為 $u = e^x F(2y - 3x)$

2. 解 $c^2u_{xx} - u_{yy} = 0$

Let $u = Ae^{\alpha x + \beta y}$, 則 $c^2\alpha^2 - \beta^2 = 0, \beta = c\alpha, -c\alpha$

$$u_1 = Ae^{\alpha(c+cy)}, u_2 = Ae^{\alpha(x-cy)}$$

General solution 為 $u = \sum_{\alpha_1} Ae^{\alpha_1(x+cy)} + \sum_{\alpha_2} Be^{\alpha_2(x-cy)} = F(x+cy) + G(x-cy)$

3. 解 $u_{xx} + u_{xy} - 2u_{yy} = x$

先求 齊次式的解 $u_c = \sum_{\beta_1} Ae^{\beta_1(x+y)} + \sum_{\beta_2} Be^{\beta_2(y-2x)} = F(x+y) + G(y-2x)$

Let 特別解 $u_p = kx^3$, 解得 $k = \frac{1}{6}$

General solution $u = F(x+y) + G(2y-x) + \frac{1}{6}x^3$

4. 解 $u_{xx} + u_{yy} = 0$

...
 $u = F(y+ix) + G(y-ix)$

解下列各偏微分方程式.

(1) $u_x + 2u_y - 3u = 0$

(3) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$

(5) $u_{xx} - 2u_{xy} + u_{yy} = 0$

5. (7) $u_{xx} - 4u_{yy} = e^{2x+y}$

(2) $u_{xx} + 3u_{xy} + 2u_{yy} = 0$

(4) $4u_{xx} - 4u_{xy} + u_{yy} = 0$

(6) $u_{xx} - u_{yy} = 12y^2$