§ Method of characteritics

對於一個一般的一階 PDE $a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y,u)$。我們希望找到一組曲線，使得沿著這些曲線，PDE 變成常微分方程，這些曲線稱為特徵曲線（characteristic curves）。

特徵方程滿足以下的常微分方程系統：

$\frac{dx}{ds}=a(x,y),\frac{dy}{ds}=b(x,y),\frac{du}{ds}=c(x,y,u)$，其中 s 是特徵曲線的參數。

1. Transport equation

$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$ (c is a constant)，subject to u(x, 0) =f(x)

$$\frac{du}{ds}=\frac{\partial u}{\partial t}\frac{dt}{ds}+\frac{\partial u}{\partial x}\frac{dx}{ds}=0 \Rightarrow \frac{dt}{ds}=1,\frac{dx}{ds}=c,\frac{du}{ds}=0$$

$\frac{dt}{ds}=1\Rightarrow t=s$，$\frac{dx}{ds}=c\Rightarrow x=cs+x_0\Rightarrow x_0=x-cs=x-ct$

$$\frac{du}{ds}=0\Rightarrow u=u(x_0)=f(x_0)=f(x-ct)$$

$\frac{du}{ds}=0$表示 u 沿參數 s 的方向為常數。

2. $\begin{cases}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0\\ u(x,0)=f(x)\end{cases}$ $u(x,\ t)= f(-x\quad \iota$

3. $\begin{cases}\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=ye^{x}\\ u(x,x)=\sin x\end{cases}$

$$\frac{dx}{ds}=1,\frac{dy}{ds}=2,\frac{du}{ds}=ye^{x}$$

$\frac{dx}{ds}=1,\frac{dy}{ds}=2\Rightarrow x=s+t, y=2s+t$ (∵ s=0 時 y=x，u=sin t)

$$\frac{du}{ds}=ye^{x}=(2s+t)e^{s+t}=2e^{t}(se^{s})+te^{t}e^{s}$$

積分 $u=2e^{t}\int se^{s}ds+te^{t}\int e^{s}ds=2e^{t}(se^{s}-e^{s})+te^{t}e^{s}+\phi(t)$

解 s，t，s=x-y，t=2x-y

s=0 時，$u(0)=(t-2)e^{t}+\phi(t)=\sin t$，$\phi(t)=\sin t-(t-2)e^{t}$

把 s=y-x，t=2x-y，s+t=x 代入化簡就會得到

$u=\sin(2x-y)+e^{x}(y-2)+(2-2x+y)e^{2x-y}$

4. $\begin{cases}\dfrac{\partial u}{\partial t}+e^{x}\dfrac{\partial u}{\partial x}=0\\ u(x,0)=\cosh(x)\end{cases}$

$$\frac{dt}{ds}=1,\frac{dx}{ds}=e^{x},\frac{du}{ds}=0$$

$$\frac{dx}{ds}=e^{x}\Rightarrow -e^{-x}=t+C$$

By initial condition u(x,0)=cosh(x) at t=0，we determine the constant C in terms of the initial position x_0 ： $-e^{-x_0}=C$

Thus，the characteritic equation becomes： $e^{-x}=e^{-x_0}-t\Rightarrow e^{-x_0}=t+e^{-x}$

$$u(t,x)=\cosh(x_0)=\frac{1}{2}(e^{x_0}+e^{-x_0})=\frac{1}{2}(t+e^{-x})+\frac{1}{2}(t+e^{-x})^{-1}$$

5. Solve $-4u_x+u_y+u=0$

$$\frac{dx}{ds}=-4,\frac{dy}{ds}=1,\frac{du}{ds}=-u$$

$$\frac{dx}{ds}+4\frac{dy}{ds}=0\Rightarrow x+4y=c$$

$$\frac{du}{ds}=-u\Rightarrow \ln|u|=-s+k=-y+\psi(c)$$

$$u=\pm e^{\psi(c)}e^{-y}:=\phi(c)e^{-y}$$

6. Solve $-2u_x+4u_y=e^{x+3y}-5u$

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}=e^{x+3y}-5u$$

$$\frac{dx}{ds}=-2,\frac{dy}{ds}=4\Rightarrow 2x+y=c$$

$\dfrac{du}{dx}=-\dfrac{1}{2}(e^{x+3y}-5u)$ ， $\dfrac{du}{dx}-\dfrac{5}{2}u=-\dfrac{1}{2}e^{x+3y}=-\dfrac{1}{2}e^{-5x+3c}$

($\dfrac{dy}{dx}+p(x)y=q(x)$ 的積分因子為 $\mu(x)=\exp(\int p(x)dx)$)

$$(e^{-\frac{5}{2}x}u)'=-\frac{1}{2}e^{-\frac{15}{2}x+3c}$$

$$e^{-\frac{5}{2}x}u=-\frac{1}{2}\times(-\frac{2}{15}e^{-\frac{15}{2}x+3c}+\psi(c))$$

$$u=\frac{1}{15}e^{-5x+3c}-\frac{1}{2}\psi(c)e^{\frac{5x}{2}}=\frac{1}{15}e^{x+3y}+\phi(2x+y)e^{\frac{5x}{2}}$$

DeepSeek 提供的另解

Characteristic coordinates are a set of variables used to simplify and solve partial differential equations (PDEs) by aligning with the characteristic curves of the equation. These curves represent paths along which the PDE reduces to an ordinary differential equation (ODE), making the problem more tractable.

The characteristic variable $\xi=2x+y$ is constant along the characteristics

To fully parameterize the plane, introduce a second coordinate $\eta=y-2x$ (varies along characeristics)

$$x=\frac{\xi-\eta}{4},y=\frac{\xi+\eta}{2}$$

The PDE leads to $8u_\eta=e^{x+3y}-5u$ (depends only on η，since ξ is constant along charateristics)，reducing it to an ODE in η 。

$$u_\xi=u_x\frac{\partial x}{\partial\xi}+u_y\frac{\partial y}{\partial\xi}=\frac{1}{4}u_x+\frac{1}{2}u_y$$

$$u_\eta=u_x\frac{\partial x}{\partial\eta}+u_y\frac{\partial y}{\partial\eta}=-\frac{1}{4}u_x+\frac{1}{2}u_y$$

$$-2u_x+4u_y=8u_\eta$$

Solve the ODE $u_\eta+\frac{5}{8}u=\frac{1}{8}e^{\frac{7\xi+5\eta}{4}}\Rightarrow u=\frac{1}{15}e^{x+3y}+e^{-5\eta/8}F(\xi)$

齊次解 $u_\eta+\frac{5}{8}u=0\Rightarrow u=e^{\frac{-5\eta}{8}}F(\xi)$

特別解 ?

7. $u_x+u_y=u^2$ for $x\in R,y>0$ and u(x,0)=g(x) for $x\in R$

$$\frac{dx}{ds}=1,\frac{dy}{ds}=1,\frac{du}{ds}=u^2$$

$$\frac{dx}{ds}=1\Rightarrow x(s)=s+t$$

$$\frac{dy}{ds}=1\Rightarrow y(s)=s$$

$\frac{du}{ds}=u^2\Rightarrow\frac{du}{u^2}=ds\Rightarrow-\frac{1}{u}=s+c$ 。At s=0，$u(0)=g(t)$ $\Rightarrow c=-\frac{1}{g(t)}$

t=x-s=x-y，

$$u(x,t)=\frac{1}{\frac{1}{g(x-y)}-y}=\frac{g(x-y)}{1-yg(x-y)}\cdots \text{ANS}$$

8. $u_x+u_y=u^2$ for y>0，and $u(x,0)=-e^{x^2}$

$$\frac{dx}{ds}=1,\frac{dy}{ds}=1,\frac{du}{ds}=u^2$$

x=s+t

y=s=x-t$\Rightarrow t=x-y$

$$\frac{du}{ds}=u^2\Rightarrow u=\frac{1}{-s-c}$$

As s=0 (initial condition) $u(0)=-e^{t^2}=\frac{1}{-c}\Rightarrow c=e^{-t^2}$

$$u(x,y)=-\frac{1}{s+c}=-\frac{1}{y+e^{-t^2}}=-\frac{1}{y+e^{-(x-y)^2}}$$

$$u(x,y)=-\frac{1}{y+e^{-(x-y)^2}}=\frac{-e^{(x-y)^2}}{1+ye^{(x-y)^2}}\cdots \text{ANS}$$

9. Solve $u_x+u_y=\frac{1}{u},x\in R,y>0$ ，$u(x,0)=\frac{1}{1+x^2},x\in R$

$$\frac{dx}{ds}=1\Rightarrow x=s+t$$

$$\frac{dy}{ds}=1\Rightarrow y=s$$

$$\frac{du}{ds}=\frac{1}{u}\Rightarrow u^2=2(s+c)$$

At s=0，x=t

$u^2=(\frac{1}{1+t^2})^2=2c$ t=x-s=x-y，s=y

$$u^2=2s+2c=2y+(\frac{1}{1+(x-y)^2})$$

Then $u(x,y)=\sqrt{2y+(\frac{1}{1+(x-y)^2})2}$ $\cdots$ANS

10. $u_x-u_y=e^u,u(x,0)=f(x)$

$$\frac{dx}{ds}=1\Rightarrow x=s+t$$

$\frac{dy}{ds} = -1 \Rightarrow y = -s$

$\frac{du}{ds} = e^u \Rightarrow -e^{-u} = s + c$

As s=0， $c = -e^{-u(0)} = -e^{-f(t)}$

$e^{-u(x,t)} = -(s - e^{-f(t)}) = y + e^{-f(x+y)}$

$u(x, y) = -\ln(y + e^{-f(x+y)})$ …ANS

11. Solve $(x-y)u_x + u_y = u, u(x,0) = x$

$\frac{dx}{dt} = x - y$ (1) $\frac{dy}{dt} = 1 \Rightarrow y = t + c$ y(0)=0$\Rightarrow y = t$ 代入(1)

$\frac{dx}{dt} = x - t$ ， $\frac{dx}{dt} - x = -t$ 兩邊同乘積分因子 e^{-t}

$\frac{d}{dt}(xe^{-t}) = -te^{-t} \Rightarrow xe^{-t} = \int -te^{-t}dt = te^{-t} + e^{-t} + c_1$

$x = t + 1 + c_1 e^t$ $x(0) = x_0 = 1 + c_1$

$x = t + 1 + (x_0 - 1)e^t = y + 1 + (x_0 - 1)e^y$ (t 用 y 代入)

$x_0 = (x - y - 1)e^{-y} + 1$

$\frac{du}{dt} = u \Rightarrow u = u_0 e^t$ ， $u(x,0) = x_0, \therefore u_0 = x_0$

$u(x, y) = [(x - y - 1)e^{-y} + 1]e^y = (x - y - 1) + e^y$

12. $xu_t - tu_x = u$ for $x \in R, t > 0$ and u(x,0)=g(x) for $x \in R$，where g is a smooth function with compact support。

A function or distribution has compact support if it is non-zero only within a compact set。

13. Solve the equation $yu_x + u_y = u, (x, y) \in R^2$ with $u(x,0) = x^2$

$\frac{dy}{ds} = 1 \Rightarrow y = s$ ， $\frac{dx}{ds} = y = s \Rightarrow x = \frac{1}{2}s^2 + x_0$ ， $\frac{du}{ds} = u \Rightarrow u = u_0 e^s$

The initial condition $u(x_0, 0) = x_0{}^2 \Rightarrow u_0 = (x - \frac{1}{2}y^2)^2, s = y$

$u(x, y) = (x - \frac{1}{2}y^2)^2 e^y$ …ANS

14. $u_x + yu_y = \frac{(x+1)^2}{x^2+1}u, u(0,y) = y$

$$\frac{dx}{ds} = 1, \frac{dy}{ds} = y, \frac{du}{ds} = \frac{(x+1)^2}{x^2+1}u$$

The PDE can be written as $\frac{du}{dx} = \frac{(x+1)^2}{x^2+1}u$ along the characteristic curves defined as： $\frac{dy}{dx} = y$

Solve the characeristic equation gives $y = Ce^x$

Solve the ODE for u： $\frac{du}{dx} = \frac{(x+1)^2}{x^2+1}u$

$$\int \frac{du}{u} = \int \frac{(x+1)^2}{x^2+1}dx \Rightarrow \ln|u| = x + \ln(x^2+1) + C \Rightarrow u = e^C(x^2+1)e^x$$

Let $e^C = C_1$ then $u(x,y) = C_1(x^2+1)e^x$

From the initial condition， $C_1 = y$ ，here C_1 is a constant，so we need to express C_1 interms of the characteristic curve $y = Ce^x$ 。

So，at x=0，y=C。Thus $C_1 = C = ye^{-x}$

$u(x,y) = ye^{-x}(x^2+1)e^x = y(x^2+1)$ ⋯ANS

15. $u_x u_y = 2xu, u(1,y) = 2y$

DeepSeek 假設 u(x,y)=yf(x)⋯ $u(x,y) = y(x^2+1)$ OK

ChatGPT 用分離變數法 解出 $u(x,y) = 2e^{\frac{k}{2}(x^2-1)} e^{\frac{2y}{k}}$ 應該是錯的。

16. Solve the equation $xu_x + u_y - 3u = -2e^y, u(0,y) = e^y$

$$xu_x + u_y = 3u - 2e^y$$

The characteristic equations for this PDE are；

$$\frac{dx}{dt} = x, \frac{dy}{dt} = 1, \frac{du}{dt} = 3u - 2e^y$$

初始曲線為 x=0，參數化初始條件為：x(0)=0，y(0)=s， $u(0) = e^s$

$\frac{dx}{ds}=x\rightarrow x(t)=c_1e^t$。代入 initial condition $c_1=0$

$x(t)$=0(沿初始曲線，特徵線無法離開x=0)，這表明初始曲線x=0是特徵線，導致解不唯一。

$\frac{dy}{dt}=1\Rightarrow y(t)=t+c_2$。y(0)=s $\Rightarrow y=t+s$

Now solve $\frac{du}{ds}=3u-2e^y=3u-2e^{t+s}$

This is a first order linear ODE。The integrating factor is e^{-3t}

$\frac{d}{dt}(ue^{-3t})=(-2e^{t+s})(e^{-3t})=-2e^{-2t+s}$

Integrate both side；

$u(t)=e^{s+t}+c_3e^{3t}$ 代入 initial condition $u(0)=e^s\Rightarrow c_3=0$

Therefor $u(t)=e^{s+t}$

代回參數 y(t)=t+s，$u(x,y)=e^y$

但由於$x(t)$=0，此解僅在x=0上成立，無法推廣到x≠0。這表明需要允許解中包含沿特徵線的任意函數。

由於初始曲線是特徵線，解可包含沿特徵線的任意函數。觀察方程結構，設通解形式為：$u(x,y)=e^y(1+C(x))$

代回原方程式可解得$C(x)=1+kx^2$

通解為$u(x,y)=e^y(1+kx^2)$ k 是任意常數。

3/25 DeepSeek 再做一次 這次(1)不用參數線法 (2)沒有最後一段話。

我再問有無其他解。The answer is $u(x,y)=e^y+e^{3y}F(xe^{-y})$ with F(0)=0 例如 $u(x,y)=e^y+x^2e^{2y}$

17. Solve the equation $u_x-u_y+u_z=zu$，$u(x,0,0)=e^x$

$\frac{dx}{dt}=1\Rightarrow x=x_0+t$ ，$\frac{dy}{dt}=-1\Rightarrow y=-t$

$\frac{dz}{dt}=1\Rightarrow z=t$ ，$\frac{du}{dt}=zu=tu$

The ODE $\frac{dz}{dt}=tu\Rightarrow u(t)=u(0)\exp(\frac{t^2}{2})$

Using the initial condition $u(x_0,0,0)=e^{x_0}$，substitute t=z and $x_0=x-z$

$u=e^{x-z}e^{z^2/2}=e^{x-z+\frac{z^2}{2}}$ $\cdots$ANS

18. Solve $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = (y - x^2)e^y$

ydx-xdy=0 has characteristics along $x^2 + y^2 = c$

the general solution to the homogeneous equation is $u_h = F(x^2 + y^2)$

try to find the particlar solution $u_p = Axe^y$

$u(x, y) = xe^y + F(x^2 + y^2)$ for any C^1 function F

19. $\cdots$