§ Consider the heat equation $u_t = u_{xx}$ on half line x>0 and t>0，with the boundary condition $u_x(0,t) = \alpha u(0,t), u_x(\infty,t) = 0$ ，for t>0

And initial condition u(x,0)=f(x)。Here，α is a constant and f is a smooth function with $f(\infty) = f_x(\infty) = 0$

Use heat kernel to construct solution。

The fundamental solution (heat kernel) for the heat equation on the entire real line is given by：$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ 。

However，since we are working on the half-line x>0，we need to modify the heat kernel to satisfy the boundary codition at x=0。

Method of images (鏡像解)：

To satisfy the boundary condition $u_x(0,t) = \alpha u(0,t)$ ，we use the method of images。

The idea is to extend the initial condition f(x) to the entire real line in such a way that the boundary condition is automatically satisfied。

We construct the solution as：

$$u(x,t) = \int_0^\infty [K(x-y,t) + K(x+y,t)] f(y) dy$$ 。

However，this form does not yet satisfy the boundary condition $u_x(0,t) = \alpha u(0,t)$ 。

To satisfy this condition，we need to adjust the kernel。

we modify the kernel as follows：

$$u(x,t) = \int_0^\infty \left[K(x-y,t) + K(x+y,t) - 2\alpha \int_0^\infty e^{-\alpha z} K(x+y+z,t)\, dz \right] f(y)\, dy$$

This modified kernel ensures that the boundary condition is satisfied。

The solution to the heat equation on the half-line with the given boundary condition is：

$$u(x,t) = \int_0^\infty \left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} + \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y)^2}{4t}} - 2\alpha \int_0^\infty e^{-\alpha z} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y+z)^2}{4t}}\, dz \right] f(y)\, dy$$

The solution can be simplified further，but the above expression gives the general form of the solution using the heat kernel and the method of images，adjusted for the boundary condition $u_x(0,t) = \alpha u(0,t)$ 。

The solution to the heat equation on the half-line with the given boundary and initial conditions is constructed using the heat kernel and the method of images，with an additional term to satisfy the boundary condition at x=0。The final solution involves an integral over the initial condition f(y) and the modified heat kernel。