Consider the heat equation：

$$\begin{cases} u_t = u_{xx}, (x,t) \in (-\infty,\infty)\times(0,\infty) \\ \quad u(x,0) = f(x), x \in R \end{cases}$$

，where f(x) is continuous and $0 \le f(x) \le 1$

(a) Show that the equation has a solution u which satisfies $\lim_{t\to 0^+} u(x,t) = f(x)$ for $x \in R$

(b) Show that there is an f(x) with $0 \le f(x) \le 1$ such that the equation has a solution u satifying $\limsup_{t\to\infty} u(0,t) = 1$ and $\liminf_{t\to\infty} u(0,t) = 0$

The fundamental solution is $G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ and

$u(x,t) = \int_{-\infty}^{\infty} f(y) \cdot \frac{1}{\sqrt{4\pi t}} \exp(-\frac{(x-y)^2}{4t}) dy$ is a particular solution。

(a)是要求證明 solution u(x,t)在$t \to 0^+$時收斂到 initial condition f(x)
對於無限區間的熱方程，我們使用傅立葉變換來求解。

定義傅立葉變換：$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx$

對熱方程兩邊取傅立葉變換，利用導數的性質：$\frac{\partial}{\partial t} \hat{u}(k,t) = -k^2 \hat{u}(k,t)$

這是一個一階常微分方程，通解為：$\hat{u}(k,t) = \hat{f}(k) e^{-k^2 t}$

其中 $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ 是 f(x)的 Fourier transform

由反傅立葉變換$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-k^2 t} e^{ikx} dx$

這是 initial function f(x)與 Gaussian kernel 的卷積：$u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)}{4t}} dy$

我們希望證明$\lim_{t\to 0^+} u(x,t) = f(x)$，這相當於證明高斯核$\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$在$t \to 0^+$時收斂到 Dirac δ-function $\delta(x-y)$。

如果 f(x)是連續函數，則卷積(convolution)積分$\lim_{t\to 0^+} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy = f(x)$

因此，當$t \to 0^+$時 u(x,t)收斂到 initial condition f(x)，即$\lim_{t\to 0^+} u(x,t) = f(x)$。

The fundamental solution to the heat equation $u_t = u_{xx}$ is given by：

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$ 。

This function satisfies the heat equation and has the property that： $\lim_{t \to 0^+} K(x,t) = \delta(x)$ ，

where $\delta(x)$ is the Dirac delta function。

The solution u(x,t) to the heat equation with the initial condition u(x,0)=f(x) can be expressed as the convolution of f(x) with the heat kernel K(x,t)：

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) f(y) dy$$ 。

Substituting the expression for K(x,t)，we get： $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy$

As $t \to 0^+$ ，the heat kernel K(x-y，t) becomes highly concentrated around y=x。This means that the integral will be dominated by the values of f(y) near y=x。

Formally，we can use the fact that the heat kernel approximates the Dirac delta function as $t \to 0^+$ ： $\lim_{t \to 0^+} K(x-y,t) = \delta(x-y)$

Therefore, the convolution integral becomes： $\lim_{t \to 0^+} u(x,t) = \int_{-\infty}^{\infty} \delta(x-y) f(y) dy = f(x)$

§ Let $K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ ， $x \in R$ ，t>0

$f(x) \in C(R)$ ，f(x+1)=f(x) and $\int_0^1 f(x) dx = 3$。Define $u(x,t) = \int_R K(x-y,t) f(y) dy$

(a) Show that $\lim_{(x,t) \to (x_0, 0^+)} u(x,t) = f(x_0)$ for each $x_0 \in R$

(b) Show that $\lim_{t \to \infty} u(0,t) = 3$

(a) $u(x,t) = \int_R K(x-y,t) f(y) dy$ ，where $K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ is the heat kernel。

The heat kernel K(x,t) has the following properties：

1. Normalization： $\int_R K(x,t)dx = 1$ for all t>0

2. Delta function behavior：As $t \to 0^+$，K(x,t) tends to the Derac delta function $\delta(x)$。This means that for any continuous function f(x)，

   $\lim_{t\to 0^+} \int_R K(x-y,t) f(y)dy = f(x)$

The function f(x) is periodic 1，implies that f(x) can be represented as a Fourier series： $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx}$ ，where c_n are the Fourier coefficients。

The Fourier transform of the heat kernel K(x,t) is $\hat{K}(\xi,t) = \exp(-4\pi^2\xi^2 t)$

This shows that the heat kernel acts as a low-pass filter in the frequency domain, attenuating high-frequency components as $t \to 0^+$

Using the properties of the heat kernel and the periodicity of $f(x)$, we can write:

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t) f(y)\, dy.$$

As $t \to 0^+$, the heat kernel $K(x-y,t)$ becomes increasingly concentrated around $y = x$, and the integral effectively samples $f(y)$ at $y = x$. Therefore, we have:

$$\lim_{t\to 0^+} u(x,t) = f(x).$$

Since this holds for any $x \in \mathbb{R}$, it follows that:

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = f(x_0).$$