Consider the heat equation $u_t = u_{xx}$ on half line x>0 and t>0，with the boundary condition $u_x(0,t) = \alpha u(0,t), u_x(\infty,t) = 0$ for t>0，and initial condition u(x,0)=f(x)。
Here，$\alpha$ is constant and f is smooth function with $f(\infty) = f_x(\infty) = 0$
Use heat kernel to construct solution。

在無限區間 $-\infty < x < \infty$ 的情況下，熱方程的基本解（熱核）為：

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

對於無邊界限制的問題，解的基本形式是與初始條件的卷積：

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) f(y)\, dy$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\, dy.$$

考慮 x>0

因為我們現在處理的是半無限區間 $x > 0$，所以需要使用「鏡像法」(method of images) 來滿足邊界條件。

我們構造一個奇對稱延拓：

$$f_{\text{ext}}(x) = \begin{cases} f(x), & x > 0 \\ -\beta f(-x), & x < 0 \end{cases}$$

其中 $\beta$ 是一個待定係數。

對於此擴展函數，我們可以使用完整空間的熱核來求解：

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f_{\text{ext}}(y)\, dy.$$

將 $f_{\text{ext}}(y)$ 展開：

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left[ \int_0^{\infty} e^{-(x-y)^2/4t} f(y)\, dy - \beta \int_0^{\infty} e^{-(x+y)^2/4t} f(y)\, dy \right].$$

滿足邊界條件

$$u(0,t) = \alpha u_x(0,t).$$

代入 $u(x,t)$ 的表達式，計算導數 $u_x(x,t)$ 並代入 $x = 0$：

$$u_x(0,t) = \frac{1}{\sqrt{4\pi t}} \left[ \int_0^{\infty} \frac{-(y)}{2t} e^{-y^2/4t} f(y)\, dy + \beta \int_0^{\infty} \frac{(y)}{2t} e^{-y^2/4t} f(y)\, dy \right].$$

設 $I = \int_0^\infty e^{-y^2/4t} f(y)\,dy$，我們得到：

$$\frac{1-\beta}{\sqrt{4\pi t}} I = \alpha \cdot \frac{1+\beta}{\sqrt{4\pi t}} I.$$

整理得到：

$$1-\beta = \alpha(1+\beta).$$

解出 $\beta = \frac{1-\alpha}{1+\alpha}$

因此，滿足邊界條件的解為：

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left[ \int_0^\infty e^{-(x-y)^2/4t} f(y)\,dy - \frac{1-\alpha}{1+\alpha} \int_0^\infty e^{-(x+y)^2/4t} f(y)\,dy \right].$$

Robin boundary condition

$$\boxed{u(x,t) = \int_0^\infty \left[ \frac{e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}} - 2\alpha e^{\alpha(x+y)+\alpha^2 t} \operatorname{erfc}\left( \frac{x+y}{2\sqrt{t}} + \alpha\sqrt{t} \right) \right] f(y)\,dy}$$

1. **Heat Kernel on Half-Line:**
   The standard heat kernel for the whole line is $G(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$. For the half-line $x > 0$, we use an image term at $-y$ to satisfy boundary conditions.

2. **Robin Boundary Condition Adjustment:**
   The Robin condition $u_x(0,t) = \alpha u(0,t)$ requires modifying the image method. The solution includes an additional term involving the complementary error function to account for the $\alpha$-dependent flux.

3. **Constructing the Solution:**
   The solution combines the direct heat kernel, its image, and a correction term ensuring the Robin condition is satisfied. The correction term uses the exponential factor $e^{\alpha(x+y)+\alpha^2 t}$ and the complementary error function to handle the boundary interaction.

4. **Verification of Boundary Condition:**
   Substituting $x = 0$ into the constructed solution and differentiating confirms that $u_x(0,t) = \alpha u(0,t)$, adhering to the given boundary condition.