§Heat Equations

1.1 heat equation on R

1.2 I/B conditions

1.3 Diffusion (1)on R (2) R^3 (3)R^n heat flow

1.4 Separation of Variables

1.5 Examples

§ 1.1 Heat Equation on R

(1) $u_t - ku_{xx} = 0$ The fundamental solution of is $G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$

(2) $\begin{cases} u_t - ku_{xx} = f(x,t), t > 0 \\ u|_{t=0} = g(x) \end{cases}$

The constant k is called the thermal diffusivity。

The particular solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)g(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)dyds$$

(3) initial/boundary value problem on an interval I in R

$\begin{cases} u_t = ku_{xx} \\ u(x,0) = \phi(x) \\ \cdots \end{cases}$ ，u satisfies certain boundary conditions

1. Dirichlet boundary condition，where the end is held at a prescribed temperature。For example，$u(a,t) = \alpha(t)$ fixes the temperature(psooibly time-varying) at the left end。
2. Neumamm boundary condition，$\frac{\partial u}{\partial x}(a,t) = \mu(t)$
3. Robin boundary condition，$\frac{\partial u}{\partial x}(a,t) + \beta(t)u(a,t) = \tau(t)$

Each end of the bar is required to satisfy one of these boundary conditions。

4. Periodic boundary conditions

$$u(a,t) = u(b,t), \frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t)$$

§ Examples

k=1，$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ 以下(1)(2)(3)都是解

(1) $u(t,x)=t+\frac{1}{2}x^2$

(2) $u(t,x)=\frac{e^{\frac{-x^2}{4t}}}{\sqrt{4\pi t}}$

(3) $u(t,x)=e^{-t+ix}=e^{-t}(\cos x+i\sin x)$

§ 1.3.1 Diffusion

Consider a liquid in which a dye(染料)is being diffused through the liquid。

The dye will move from higher concentration to lower concentration。

Let u(x,t) be the concentration (濃度 mass per unit length) of the dye at positin x in the pipe at time t。

The total mass of dye in the pipe from x_0 to x_1 at time t is given by

$M(t)=\int_{x_0}^{x_1}u(x,t)dx$，therefore $\frac{dM}{dt}=\int_{x_0}^{x_1}u_t(x,t)dx$

By Fick' s law，$\frac{dM}{dt}=$ flow in － flow out= $ku_x(x_1,t)-ku_x(x_0,t)$，where k>0

$\int_{x_0}^{x_1}u_t(x,t)dx=ku_t(x_1,t)-ku_t(x_o,t)$，differentiating with repect to x_1，we have

$u_t(x_1,t)=ku_{xx}(x_1,t)$ or $u_t=ku_{xx}$

§ 1.3.2 推導 Heat equation $\frac{\partial u}{\partial t}=\Delta u$ Jean le Rond d' Alembert 1746

A body occupying a volume R with surface S

Heat capacity c

Density of matter ρ

Absolute temperature T posseses a source of heat of intensity q

$Q=\iiint_R \rho cTdxdydz$ the amount of heat Q inside R

$V=-KgradT$，K>0

$\frac{dQ}{dt}=-\int_S V_n d\Sigma+\int_R qdxdydz$，the amount of heat passing through S in unit time

$$V_n = -K\frac{\partial T}{\partial n}$$

$$\iiint_R \frac{\partial}{\partial t}(\rho cT)dxdydz = \iint_S K\frac{\partial T}{\partial n}d\Sigma + \iiint_R qdxdydz$$ ，

where $\iint_S K\frac{\partial T}{\partial n}d\Sigma = \iiint_R -divVdxdydz$ (divergence theorem)

$$\frac{\partial}{\partial t}(\rho cT) = q - divV = q + \frac{\partial}{\partial x}(K\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(K\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(K\frac{\partial T}{\partial z})$$

$\frac{\partial T}{\partial t} = a^2\Delta T$ ，if ρ, c, K are constant and q=0

§ 1.3.3 heat flow

D is a region in R^n ， $x = (x_1, x_2, ..., x_n)$ is a vector。

u(x,t) is the temperature at point x，time t
Let H(t) be the total amount of heat contained in D，
c be the specific heat of the material， ρ its desity of the material。

$$H(t) = \int_D c\rho u(x,t)dx$$

Fourier’ s law：heats flows from hot to cold region at a rate κ>0 proportional to the temperature gradient。

$u(x,t) = -\kappa(x)\frac{\partial u}{\partial t}$ is known as Fourier’ s Law of Cooling。

$\kappa(x) > 0$ is the thermal conductivity of the bar at position x。
The only way heat will leave D is through the boundary。

$$\frac{dH}{dt} = \int_D c\rho u_t(x,t)dx = \int_{\partial D} \kappa\nabla u \cdot ndS$$

Where n is the outward unit vector to ∂D
dS：surface measure over ∂D
Divergence theorem：

$$\int_{\partial D} F \cdot ndS = \int_D \nabla \cdot Fdx$$

$$\therefore \int_{\partial D} c\rho u_t(x,t)dx = \int \nabla \cdot (\kappa\nabla u)dx$$

$c\rho u_t = \nabla \cdot (\kappa\nabla u)$ ， $u_t = k\Delta u$ where $k = \frac{\kappa}{c\rho} > 0, \Delta u = \sum_i u_{x_i x_i}$

§ 1.4 Separation of Variables

$u_t - ku_{xx} = 0$

Let u(x,t)=X(x)T(t)，then $XT' - kX''T = 0$ ，$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$

$\frac{T'}{kT} = -\lambda, \frac{X''}{X} = -\lambda$ ，稱為 eigenvalue problem

§ 1.2.1

Example 1 Dirichlet boundary coditions

$$\begin{cases} X'' = -\lambda X & 0 < x < l \\ X(0) = X(l) = 0 \end{cases}$$

$X'' = -\lambda X$ 分別對$(1)\lambda > 0\,(2)\lambda = 0\,(3)\lambda < 0$ 討論

(1) $\lambda = \beta^2 > 0, then\, X(x) = C\cos\beta x + D\sin\beta x$

$X(0) = 0 \Rightarrow C = 0$

$X(l) = 0 \Rightarrow \sin\beta l = 0, \beta = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$

$We\, have\, \lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n = D_n \sin\left(\frac{n\pi x}{l}\right)$

(2) For $\lambda = 0\, or\, \lambda < 0$ there are no eigenvalues。

$u_n(x,t) = T_n(t)X_n(x) = Ae^{-k\lambda t}\left(D_n \sin\frac{n\pi x}{l}\right)$

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ is the solution which satisfies the boundary condition

Example 2

(Periodic Boundary Conditions) Find all solutions to the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & -l < x < l \\ X(-l) = X(l), \;\; X'(-l) = X'(l). \end{cases} \tag{2.5}$$

The solutions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = C_n \cos\left(\frac{n\pi}{l}x\right) + D_n \sin\left(\frac{n\pi}{l}x\right) \qquad n = 1, 2, \ldots$$

$$\lambda_0 = 0 \qquad X_0(x) = C_0.$$

§ 1.2.2 For initial conditions

Cauchy problem for $\begin{cases} u_t = a^2 u_{xx} \\ u_{t=0} = \varphi(x)\, is\, a\, contuous\, bounded\, function \end{cases}$

Then $u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) \exp\left(-\frac{(\xi - x)^2}{4a^2 t}\right) d\xi$ for t>0

$\lim_{t \to 0} u(x,t) = \varphi(x)$

以下用分離變數法

$X = A\cos\lambda x + B\sin\lambda x, T = \exp\left(-a^2\lambda^2 t\right)$

$u_\lambda(x,t) = \exp\left(-a^2\lambda^2 t\right)(A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)$ is a solution

So is $\int_{-\infty}^{\infty} u_\lambda(x,t), \varphi(x) = \int_{-\infty}^{\infty}(A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)d\lambda$

Since $\varphi(x)$ is ontinuous and bounded，it gas a Fourier Integral representarion …

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(2) $\begin{cases} u_t - ku_{xx} = f(x,t), t > 0 \\ u\big|_{t=0} = g(x) \end{cases}$

The particular solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)g(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)dyds$$

§ 1.5 Example Justin Ko W5

1. $\begin{cases} u_t = ku_{xx} & t > 0 \\ u(x,0) = x \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(y-x)^2}{4kt}} y dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} (y-x) e^{\frac{-(y-x)^2}{4kt}} dy + \frac{x}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-\left(y-x0^2\right)}{4kt}} dy = x$$

前一項 奇函數積分=0。

Normal distribution

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is the probability density function

後一項 積分值=x

2. $\begin{cases} u_t = ku_{xx} & t > 0 \\ u(x,0) = x^2 \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(y-x)^2}{4kt}} y^2 dy$$

Let $p = \dfrac{y-x}{\sqrt{4kt}}, dp = \dfrac{dy}{\sqrt{4kt}}$, $\dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp = 1, \int_{-\infty}^{\infty} p e^{-p^2} dp = 0$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(x + \sqrt{4kt}p\right)^2 e^{-p^2} dp = \ldots = x^2 + kt$$

3. 解 $\begin{cases} u_t - ku_{xx} = x^2, t > 0 \\ u\big|_{t=0} = 0 \end{cases}$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \exp\left(\frac{-(x-y)^2}{4\pi(t-s)}\right) y^2 dy ds$$

$$\frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(y-x)^2}{4k(t-s)}} y^2 dy = x^2 + 2k(t-s)$$

$$u(x,t) = \int_0^t x^2 + 2k(t-s) ds = x^2 t + kt^2$$

4.