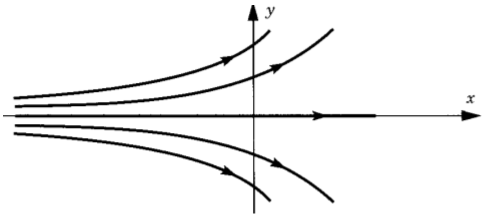
§ First order semilinear equations

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$ …characteristic equation

The method of characteristics by Justin Ko Week2 p.11~12

§ 解釋 $u_x + yu_y = 0$…(1) Walter A. Strauss p.8



Itself asserts that the directional derivative in the direction of the vector (1, y) is zero。The curves in the xy plane with (1, y) as tangent vectors have slop y。Their equations are $\frac{dy}{dx} = \frac{y}{1}$

The ODE has the solution $y = Ce^x$

These curves are called the characteristic curves of the PDE (1)

As C is changed，the curves fill out the xy plane perfectly without intersecting。

On each of the curves u(x,y) is a constant because

$$\frac{d}{dx}u(x, Ce^x) = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = u_x + yu_y = 0.$$

Thus，$u(x, Ce^x) = u(0, Ce^0) = u(0, C)$ is independent of x。

Putting $y = Ce^x$ and $C = e^{-x}y$ ，we have $u(x,y) = u(0, e^{-x}y)$

It follows that $u(x,y) = f(e^{-x}y)$ is the general solution of this PDE。

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

曲面 z=z(x,y)，$(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1)$是曲面的法向量

z=f(x,y)，X=(x,y,f(x,y)) then $X_x = (1,0,\frac{\partial f}{\partial x}), X_y = (0,1,\frac{\partial f}{\partial y})$ ，

$X_x \times X_y = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$ …微分幾何是這麼做的

則(P,Q,R)與曲面的法向量垂直，所以落在切平面上，在切線方向。在曲面上的點，<dx,dy,dz>在通過該點的切線方向，所以$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

上式為兩聯立常微分方程 設其解為

$$\begin{cases} u(x,y,z)=c_1 \\ v(x,y,z)=c_2 \end{cases}$$

原方程式的解為 F(u,v)=0

1. 解 $x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=(x+y)z$

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}$$

$$-\frac{1}{x}+\frac{1}{y}=c_1$$

$$\frac{dx-dy}{x^2-y^2}=\frac{dz}{(x+y)z}\ ,\ \frac{d(x-y)}{x-y}=\frac{dz}{z}\Rightarrow\frac{x-y}{z}=c_2$$

General solution 為 $F(\dfrac{x-y}{xy},\dfrac{x-y}{z})=0$

又 $\dfrac{z}{xy}=\dfrac{c_1}{c_2}=c_3$，general solution 為 $F_2(\dfrac{x-y}{z},\dfrac{z}{xy})=0$

2. 解 $(x^2-y^2-z^2)\dfrac{\partial z}{\partial x}+2xy\dfrac{\partial z}{\partial y}=2xz$

3. 解 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}+t\dfrac{\partial z}{\partial t}=xyt$ $\qquad F(\dfrac{x}{y},\dfrac{t}{y},xyt-3z=)$

§ 解 $y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=1$

$$\frac{dx}{y}=\frac{dy}{-x}=\frac{du}{1}$$

$$\frac{dx}{y}=\frac{dy}{-x}\Rightarrow x^2+y^2=c$$

$$\frac{dy}{-x}=\frac{du}{1}\Rightarrow\frac{dy}{-\sqrt{c-y^2}}=\frac{du}{1}\Rightarrow u=-\int\frac{dy}{\sqrt{c-y^2}}$$

$let\,y=\sqrt{c}t,\sqrt{c-y^2}=\sqrt{c}\sqrt{1-t^2}$

$$\int\frac{dy}{\sqrt{c-y^2}}=\int\frac{dt}{\sqrt{1-t^2}}=\sin^{-1}t=\sin^{-1}\frac{y}{\sqrt{c}}=\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}=\tan^{-1}\frac{y}{x}$$

$$u(x,y)=-\tan^{-1}\frac{y}{x}+F(c)=-\tan^{-1}\frac{y}{x}+F\left(x^2+y^2\right)$$

Examples

1. 解 $\begin{cases} u_x + xu_y = 0, x, y \in R \\ u\big|_{x=0} = \sin y, y \in R \end{cases}$ $u(x,y) = \sin\left(y - \frac{x^2}{2}\right)$

2. 解 $\begin{cases} u_x + xu_y = 0 \\ u\big|_{y=0} = f(x) \end{cases}$

3. 解 $xu_x + yu_y = xe^{-u}$ with u(x,y)=0 on $y = x^2$

$\frac{dy}{dx} + p(x)y = q(x)$ has an integration factor $\mu(x) = \exp\left(\int p(x)dx\right)$

$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}$

$\frac{dy}{dx} - \frac{1}{x}y = 0$ has an integration factor $\mu(x) = \exp\left(\int \frac{-1}{x}\right)dx = \frac{1}{x}$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} = c$,...

4. 解 $2xyu_x + (x^2 + y^2)u_y = 0$ with $u(x,y) = \exp(\frac{x}{x-y})$ on x+y=1

$\frac{dx}{2xy} = \frac{dy}{x^2+y^2} \Rightarrow (x^2+y^2)dx - 2xydy = 0$

$M = x^2 + y^2, N = -2xy \Rightarrow \frac{M_y - N_x}{N} = -\frac{2}{x}$

The integration factor is $\mu(x) = \exp\int \frac{-2}{x}dx = x^{-2}$ 可解出

$x - \frac{y^2}{x} = c \Rightarrow u(x,y) = f\left(\frac{x^2-y^2}{x}\right)$

On x+y=1, $u(x,y) = f\left(\frac{x-y}{x}\right) = \exp\left(\frac{x}{x-y}\right) \Rightarrow f(\xi) = \exp\left(\frac{1}{\xi}\right)$

Thus, $u(x,y) = \exp\left(\frac{x}{x^2-y^2}\right)$

5. 解 $u_x + 3u_y - 2u_z = u$ with u(0,y,z)=f(y,z)

6. 解 $yu_x + xu_y + u_z = 0$ with u(x,y,0)=f(x,y)

Walter A. Strauss p.10

EXERCISES

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

3. Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.
4. Check that (7) indeed solves (4).
5. Solve the equation $xu_x + yu_y = 0$.
6. Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.
7. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.
   (b) In which region of the xy plane is the solution uniquely determined?
8. Solve $au_x + bu_y + cu = 0$.

9. Solve the equation $u_x + u_y = 1$.
10. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.
11. Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f\, ds + g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (*Hint:* Use the coordinate method.)
12. Show that the new coordinate axes defined by (3) are orthogonal.
13. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

6. $u(x,y) = y - \sin^{-1} x$

7. (a) $u(x,y) = e^{x^2-y^2}$ (b)

8. $u(x,y) = \exp\left(-\frac{c}{a}x + \phi(bx - ay)\right)$

$u(x,y) = \exp\left(-c(ax+by)/(a^2+b^2)\right) f(bx - ay)$

9. $u(x,y) = x + \phi(x-y)$

10. $u(x,y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}$

11. Coordinate method ：

Let $\xi = ax + b, \eta = bx - a$, then

$u_x = au_\xi + bu\eta, u_y = bu_\xi - au_\eta$

$au_x + bu_y = (a^2 + b^2)u_\xi$

Thus the solution is $u = f(\eta) = f(bx - ay)$

???

12.

1. 解下列各偏微分方程式.

(1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

(2) $y^2z\frac{\partial z}{\partial x} - x^2z\frac{\partial z}{\partial y} = x^2y$

(3) $(y-z)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = z - x$

(4) $(x^2 - y^2 - z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz$

(5) $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0$，$a$ 與 b 皆不為 0

2. 解 $(x-a)\frac{\partial z}{\partial x} + (y-b)\frac{\partial z}{\partial y} = z - c$， 並求滿足此偏微分方程式且通過 xy-平面上曲線 $x^2+y^2=r^2$ 的曲面方程式，其中 a、b、c 及 r 皆為常數.

3. 求滿足 $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0$ 且通過平面 $z=1$ 上雙曲線 $xy=x+y$ 的曲面方程式.

4. 解 $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0.$