§ Linear and nonlinear Waves Peter J. Olver Ch2

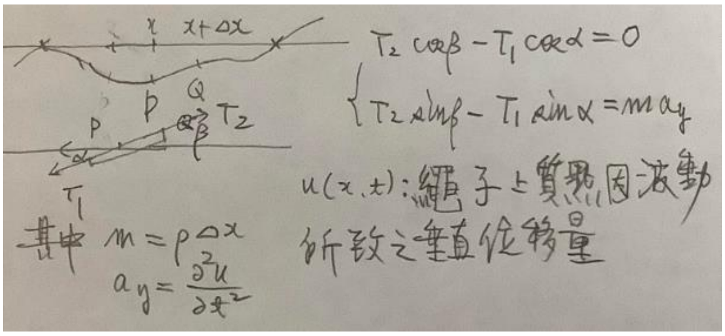
2.4 The Wave Equations

External Forcing and Resonance(共鳴 諧振)

波動方程(wave equation)

§ vibrating string

導出琴弦的波動方程式 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ Brook Taylor 1714



1. 設 string 單位長之質量 $=\rho\, {}^{kg}\!/\!_{m}$ 。
2. 兩端之張力夠大，重力可以省略。
3. 垂直位移量很小，每一質點都在同一垂直線上移動，且 string 的斜率很小

$\alpha \approx \beta \approx 0$ ， $\cos\alpha \approx \cos\beta \approx 1$

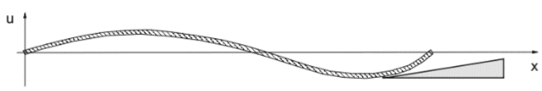
$\sin\alpha \approx \tan\alpha = \frac{\partial u(x,t)}{\partial x}$ ， $\sin\beta \approx \tan\beta = \frac{\partial u(x+\Delta x,t)}{\partial x}$

$$\begin{cases} T_2 - T_1 = 0 \\ T_2 \frac{\partial u(x+\Delta x,t)}{\partial x} - T_1 \frac{\partial u(x,t)}{\partial x} = \rho \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} \end{cases}$$

$$T \frac{\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}}{\Delta x} = \rho \frac{\partial^2 u(x,t)}{\partial^2 t}$$

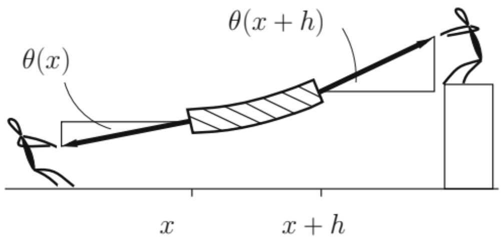
Let $\Delta x \to 0$ ， $\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$ ， $c = \sqrt{\frac{T}{\rho}}$ ：波速

牛頓=kg.m.s^-2



A string displacement is shown at one value of time。

The displacement and slop of the string are assumed very small everywhere。



Tension forces depend on angles $\theta(x),\theta(x+h)$ at the ends of a bit of the string。

$$\frac{\partial u}{\partial x}=\tan\theta(x)=\frac{\sin\theta}{\cos\theta}\approx\theta$$

The forces in the (x,u)plane acting on the piece of the string at x and x+h are
$-T(\cos\theta(x),\sin\theta(x)),T(\cos\theta(x+h),\sin\theta(x+h))$
We consider the motion of the string only in the u direction。

$$\text{Total force}=T\sin\theta(x+h)-T\sin\theta(x)=hT\cos\theta\frac{d\theta}{dx}=hT\frac{\partial^2 u}{\partial x^2}$$

$$\lim_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h}=\cos\theta\text{ ， }\cos\theta\approx 1,\theta=\frac{\partial u}{\partial x}$$

Total force=mass times acceleration= hWu_{tt} ，where hW is th mass。

$hT\frac{\partial^2 u}{\partial x^2}=hWu_{tt}$ ，let $c=\sqrt{\frac{\mathrm{T}}{\mathrm{W}}}$ ，then we get $u_{tt}-c^2u_{xx}=0$

§ Solution

(1) 齊次 $u_{tt}-c^2u_{xx}=0$

(2) $\begin{cases} u_{tt}-c^2u_{xx}=f(x,t),x\in R,t>0 \\ u\big|_{t=0}=g(x),x\in R \\ u_t\big|_{t=0}=h(x),x\in R \end{cases}$

(1) 的一般解 $u(t,x)=\phi(x-ct)+\psi(x+ct)$
Brook Taylor 1714 年
(2) 的特別解

$$u(t,x)=\frac{g(x+ct)+g(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}h(s)ds+\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}f(y,s)dyds$$

By $d'Alembert$ 1746 年

it is the superposition of two waves moving with speed c in opposite directions.

The information provided by the initial data propagates along the *characteristics*

$$x\pm ct=\text{constant}.$$

In particular, the solution at the point (x,t) depends only on the value of h on the entire interval $[x-ct,x+ct]$ and those of g at the endpoints.

• *Global Cauchy problem* ($n \geq 2$). In dimension $n \geq 2$ the global Cauchy problem reads

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

If $n = 3$, $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$, then the only C^2 solution on $\mathbb{R}^3 \times [0, +\infty)$ is provided by *Kirchhoff's formula*

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \int_{\{|\mathbf{x}-\boldsymbol{\sigma}|=ct\}} g(\boldsymbol{\sigma})\, d\sigma\right] + \frac{1}{4\pi c^2 t} \int_{\{|\mathbf{x}-\boldsymbol{\sigma}|=ct\}} h(\boldsymbol{\sigma})\, d\sigma.$$

In case $n = 2$, $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$, the only C^2 solution on $\mathbb{R}^2 \times [0, +\infty)$ is determined by *Poisson's formula*

$$u(\mathbf{x}, t) = \frac{1}{2\pi c}\left\{\frac{\partial}{\partial t} \int_{\{|\mathbf{x}-\mathbf{y}| \leq ct\}} \frac{g(\mathbf{y})\, d\mathbf{y}}{\sqrt{c^2 t^2 - |\mathbf{x}-\mathbf{y}|^2}} + \int_{\{|\mathbf{x}-\mathbf{y}| \leq ct\}} \frac{h(\mathbf{y})\, d\mathbf{y}}{\sqrt{c^2 t^2 - |\mathbf{x}-\mathbf{y}|^2}}\right\}.$$

• *Domain dependence.* For $n = 3$, Kirchhoff's formula shows that $u(\mathbf{x}, t)$ depends only on the values of the data assumed on the *sphere*

$$\{\boldsymbol{\sigma} \in \mathbb{R}^3 : |\mathbf{x} - \boldsymbol{\sigma}| = ct\}.$$

In dimension $n = 2$, the solution at $(\mathbf{x}, t)$ depends on the values of the data assumed on the disc

$$\{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| \leq ct\}.$$

[大自然的數學遊戲 p.86] Ian Stewart 1945-

琴弦，決定音調的是振動的頻率。

1714 年 Brook Taylor 發表小提琴的基本振動頻率公式，它完全由琴弦的長度，拉力與密度決定。

1746 年 Jean Le Rond *d'Alembert* 證明了琴弦的許多種振動都不是正弦式駐波。

1748 年 Leonhard Euler 推導出琴弦的波動方程式。

琴弦每一小段的加速度都與這一小段所受的拉力成正比。

Euler 不僅導出波動方程式，還求出它的解。

Daniel Bernoulli(1700-1782)也解出了波動方程式，不過用的是不同的方法。

根據伯努利的理論，最一般的解可以表示為無限多個正弦波的疊合。

這裡開啟一世紀的爭論。P.92

1759 年 Euler 開始研究鼓面的問題，並且導出一個波動方程式，它能描述鼓面在垂直方向的位移如何隨時間變動。

§ 電磁波

Michael Faraday 1791-1867

James Clerk Maxwell 1831-1879

Heinrich Hertz 1857-1894 1887 無線電

§ 獵戶星座α 星(參宿四 Betelgeuse)

Walter A. Strauss

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0.$$

For a transport equation $u_t + cu_x = 0$ ，the general solution is $u(t,x) = \phi(x-ct)$

Let $v = u_t + cu_x$ then

$$\begin{cases} v_t - cv_x = 0 \Rightarrow v(x,t) = h(x+ct) \\ u_t + cu_x = v = h(x+ct) \end{cases}$$

the homogeneous solution of $u_t + cu_x = 0 \Rightarrow u(x,t) = g(x-ct)$

for a particular solution,let u(x,t)=f(x+ct) then

$u_t + cu_x = c\dfrac{\partial f}{\partial \xi} + c\dfrac{\partial f}{\partial \xi}, where\, \xi = x + ct = h(x-ct)$, $f'(s) = \dfrac{h(s)}{2c}$

一般解=其次解+特別解

u(x,t)=g(x-ct)+f(x+ct)

另一個解法是令 $\xi = x + ct, \eta = x - ct$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta} \end{cases}$$

寫成$\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}, \dfrac{\partial}{\partial t} = c\dfrac{\partial}{\partial \xi} - c\dfrac{\partial}{\partial \eta}$

$\partial_t - c\partial_x = -2c\partial_\eta, \partial_t + c\partial_x = 2c\partial_\xi$

$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_\xi)(2c\partial_\eta)u = u_{tt} - c^2 u_{xx} = 0$

Which means that $u_{\xi\eta} = 0$

The solution of the transformed equation is $u = f(\xi) + g(\eta)$

The wave equation has a nice simple geometry. There are *two* families of characteristic lines, $x \pm ct =$ constant, as indicated in Figure 1. The most general solution is the sum of two functions. One, $g(x - ct)$, is a wave of arbitrary shape traveling to the *right* at speed c. The other, $f(x + ct)$, is another shape traveling to the *left* at speed c. A "movie" of $g(x - ct)$ is sketched in Figure 1 of Section 1.3.

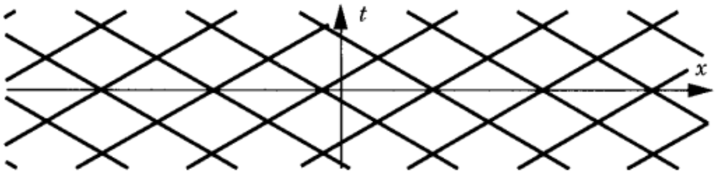


Figure 1

§ 變數分離法

1. $\begin{cases} u_{tt} = c^2 u_{xx}, 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, t \geq 0 \\ u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq l \end{cases}$

設 u(x,t)=X(x)T(t) 則 $u_{tt} = X(x)T''(t), u_{xx} = X''(x)T(t)$

$X(x)T''(t) = c^2 X''(x)T(t)$ or $\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = \lambda$

$X'' = \lambda X, T'' = \lambda c^2 T$ 反映周期函數時 $\lambda < 0$ ，令 $\lambda = -\omega^2$

$X = A\cos\omega x + B\sin\omega x$

$T = C\cos c\omega t + D\sin c\omega t$

$u(x,t) = X(x)T(t) = (A\cos\omega x + B\sin\omega x)(C\cos c\omega t + D\sin c\omega t)$

因為 $T(t) \neq 0$

所以 $X(0) = A = 0, X(l) = B\sin\omega l = 0$ ， $B \neq 0$

$\omega = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$

$u_n(x,t) = (C_n \cos\frac{nc\pi}{l}t + D_n \sin\frac{nc\pi}{l}t)\sin\frac{n\pi x}{l}$ ，n=1,2,3,...

$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}(C_n \cos\frac{nc\pi}{l}t + D_n \sin\frac{nc\pi}{l}t)\sin\frac{n\pi x}{l}$

$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x) \Rightarrow C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ n=1,2,3,...

$u_t(x,0) = \sum_{n=1}^{\infty} D_n \times \frac{nc\pi}{l}\sin\frac{n\pi x}{l} = g(x) \Rightarrow D_n = \frac{2}{nc\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$ n=1,2,3,...

u(0,t)=u(l,t)=0 means that the string with tied ends。

Suppose u is of the form u(x,t)=f(x-ct)+g(c+ct) and satisfies the boundary condition。

The boundary conditions state that f(-ct)+g(ct)=0， $f(l-ct) + g(l+ct) = 0$

Denoting ct as x, the first relation says that $f(-x) = -g(x)$. Setting this into the second relation we get

$$-g(x-a) + g(x+a) = 0.$$

Denoting $x-a$ as y we rewrite this as

$$g(y+2a) = g(y).$$

In words: g is a periodic function with period $2a$.

Since $f(y) = -g(-y)$, it follows that also f is periodic with period $2a$. Therefore $u(x,t) = f(x-ct) + g(x+ct)$ is a periodic function of t with period $\frac{2a}{c}$.

A function with period p also has periods $2p$, $3p$, and so forth. Thus a string that vibrates with period $\frac{2a}{c}$ also vibrates with period $\frac{2an}{c}$, n any whole number.

2. 以下都是$u_{tt} = 4u_{xx}$的解

(1) cos(x-2t)

(2) e^{x+2t}

(3) $x^2 + 2xt + 4t^2$

(4) $4t^2 - x^2$

(5) Cos(x+2t)

(6) Sin2tcosx

(7) $e^{-(x-2t)^2}$


Initial-value problem ：Walter A. Strauss

$u_{tt} = c^2 u_{xx}, -\infty < x < \infty$

$u(x,0) = \phi(x)$

$u_t(x,0) = \psi(x)$

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

Is a bona fide(真實不虛的) solution of this IVP。

Due to $d'Alembert$ 1746

Examples

1. $u_{tt} = c^2 u_{xx}, -\infty < x < \infty$

$u(x,0) = 0$

$u_t(x,0) = \cos x$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \cos x dx = \frac{1}{c}\cos x \sin ct$$

2. The plucked string (撥動琴弦的方程式)　　p.36

3. Pinched string(捏緊的繩子)

A guitar (initially at rest)is pinchedat the midpoint and releared。
Denoting the string density by ρ and the tension by τ，formulate the mathematical model and write the solution as superposition of standing wave。

Solution. Let L be the string length and suppose that the string at rest lies along the x-axis between 0 and L. Denote by $u(x,t)$ the displacement from the rest position of the point x at time t, and let a be the initial displacement of $x = L/2$. The initial configuration of the string, once it is pinched in the middle, is described by the function

$$g(x) = a - \frac{2a}{L}\left|x - \frac{L}{2}\right| = \begin{cases} 2ax/L & 0 \le x \le L/2 \\ 2a(L-x)/L & L/2 \le x \le L. \end{cases}$$

If a is small with respect to the length and we ignore the string weight, u solves

$$u_{tt} - c^2 u_{xx} = 0$$

where $c = \sqrt{\tau/\rho}$ is the travelling speed of waves along the string. The fixed endpoints impose homogeneous Dirichlet conditions at the boundary of the interval, while the initial rest status means that the initial velocity is zero. All this gives the following model:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0 & t \ge 0 \\ u(x,0) = g(x),\ u_t(x,0) = 0 & 0 \le x \le L. \end{cases}$$

PDE in Action by Gianmaria Verzini p.219

$$u(x,t) = \frac{8a}{\pi^2}\sum_{h=0}^{+\infty} \frac{(-1)^h}{(2h+1)^2} \cos\left(\frac{c\pi(2h+1)}{L}t\right) \sin\left(\frac{(2h+1)\pi}{L}x\right).$$

4.

Problem 4.2.2 (Reflection of waves). *Consider the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x),\ \ u_t(x,0) = 0 & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & t \ge 0. \end{cases}$$

a) *Define suitably the datum g outside the interval $[0, L]$, and use d'Alembert's formula to represent the solution as superposition of traveling waves.*

b) *Examine the physical meaning of the result and the relationship with the method of separation of variables.*

5.

Problem 4.2.3 (Equipartition of energy). *Let u denote the solution to the following global Cauchy problem for the vibrating string:*

$$\begin{cases} \rho u_{tt} - \tau u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

Assume g and h are regular functions that vanish outside a compact interval $[a, b]$. Prove that after a sufficiently long time T

$$E_{cin}(t) = E_{pot}(t) \qquad \text{for any } t \ge T.$$

6.

Problem 4.2.4 (Global Cauchy problem – impulses). *Find the formal solution to the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

with the following initial data:

a) $g(x) = 1$ *if* $|x| < a$, $g(x) = 0$ *if* $|x| > a$; $h(x) = 0$.

b) $g(x) = 0$; $h(x) = 1$ *if* $|x| < a$, $h(x) = 0$ *if* $|x| > a$.

Problem 4.2.5 (Forced vibrations). *Consider the problem*

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & 0 < x < L,\ t > 0 \\ u(x,0) = u_t(x,0) = 0 & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & t \ge 0, \end{cases} \tag{4.9}$$

with $f \in C^3([0,L] \times [0,+\infty))$ *and* $f(0,t) = f(L,t) = f_{xx}(0,t) = f_{xx}(L,t)$ *for any* $t \ge 0$.

a) *Solve the problem by the separation of variables. Show that the expression found is the only classical solution.*

b) *Study in detail the case*

$$f(x,t) = g(t) \sin\left(\frac{\pi x}{L}\right).$$

7.

Problem 4.2.6 (Semi-infinite string with fixed end). *Consider the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0,\ t > 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & x \ge 0 \\ u(0,t) = 0 & t \ge 0, \end{cases}$$

with g, h *regular,* $g(0) = 0$.

a) *Extend suitably the initial data to* $\mathbb{R}$ *and use d'Alembert's formula to write a representation formula for the solution.*

b) *Interpret the solution in the case* $h(x) = 0$ *and*

$$g(x) = \begin{cases} \cos(x-4) & |x-4| \le \dfrac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

8.

Problem 4.2.7 (Forced vibrations of a semi-infinite string). *A semi-infinite string is initially at rest along the axis* $x \ge 0$, *and fixed at* $x = 0$. *An external force* $f = f(t)$ *sets it in motion.*

a) *Write the mathematical model governing the vibrations.*

b) *Solve the problem using the Laplace transform in* t, *assuming that the transform of* u *is bounded as* s *tends to* $+\infty$.

9.

10.

Problem 4.2.8 (Vibrations of a hanging chain). *In this problem we shall find the equation governing the small (plane) vibrations of a hanging chain of length L. Call $u = u(x,t)$ the displacement from the horizontal position and ρ the linear density of mass (a constant). Let us assume that the chain is completely flexible (that is, no resistance to deformations) and that the oscillations are only transverse (the chain moves on a vertical plane).*

a) *Denote by $\boldsymbol{\tau}(x+\Delta x)$ and $\boldsymbol{\tau}(x)$ the tensions at points $x+\Delta x$ and x relatively to some small interval $(x, x+\Delta x)$ on the chain; these tensions are the forces acting on that portion of chain from below and above respectively. Argue as for the vibrating string and show that, up to first order approximation,*

$$|\boldsymbol{\tau}(x)| = \tau(x) = \rho g x$$

(where g is the acceleration of gravity).

b) *Show that small vibrations are governed by the equation*

$$u_{tt} = g(xu_{xx} + u_x).$$

11.

Problem 4.2.9 (Hanging chain – separation of variables). *In relation to the previous problem solve (by separation of variables)*

$$\begin{cases} u_{tt} = g(xu_{xx} + u_x) & 0 < x < L,\ t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = h(x) & 0 \le x \le L \\ u(L,t) = 0,\ |u(0,t)| \text{ bounded} & t \ge 0. \end{cases}$$

12.

Problem 4.2.10 (Sound waves in a pipe). *Let P_1, P_2 be two identical, cylindrical organ pipes of length L. Assume that its axis is the segment $[0, L]$ along the z-direction. Pipe P_1 is stopped (closed) at $z = 0$ and open at $z = L$, whereas P_2 is open at both ends.*

Pressing a key makes pressurised air move through the pipes. Which pipe produces the note of higher pitch (i.e., higher frequency)?

13. 解 $\begin{cases} u_{tt} - 4u_{xx} = 0, t > 0 \\ u\big|_{t=0} = \tanh(x), x \in R \\ u_t\big|_{t=0} = \arctan(x), x \in R \end{cases}$

$$u(x,t) = \frac{\tanh(x+2t) + \tanh(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)dy$$

$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}, \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$

$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

Where $u = \tan^{-1} x, v = x, \int u dv = uv - \int v du$ ，$\frac{du}{dx} = \frac{1}{1+x^2}$

$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \frac{1}{2} \ln \left|1+x^2\right| + c$

Then $\frac{1}{4} \int_{x-2t}^{x+2t} \tan^{-1} y dy = \frac{1}{4} (y \tan^{-1} y - \frac{1}{2} \ln \left|1+y^2\right|)_{y=x-2t}^{x+2t}$ =...

14. 解 $\begin{cases} u_{tt} - 4u_{xx} = f(x,t), t > 0 \\ u\big|_{t=0} = g(x) \\ u_t\big|_{t=0} = h(x) \end{cases}$ ，$f(x,t) = \begin{cases} \sin x, 0 < t < \pi \\ 0, t \ge \pi \end{cases}$

15. Find a solution of each problem in the form u(x,t)=f(x-ct)+g(x+ct)

(a) $u_{tt} - c^2 u_{xx} = 0$ with $u(x,0) = \sin x$ and $u_t(x,0) = 0$.
(b) $u_{tt} - u_{xx} = 0$ with $u(x,0) = 0$ and $u_t(x,0) = \cos(2x)$.
(c) $u_{tt} - 25u_{xx} = 0$ with $u(x,0) = 3\sin x + \sin(3x)$ and $u_t(x,0) = \cos(2x)$.

16. Verify that every function of the form u(x,t)=f(x-ct)+g(x+ct) where f and g are twice differentiable functions of a single variable，is a solution of

$u_{tt} - c^2 u_{xx} = 0$

Exerises 2.1 The Wave Equation Walter A. Strauss

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

$u_{tt} = c^2 u_{xx}, -\infty < x < \infty \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)$

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds.$$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ the solution is $u(x,t) = e^x \cosh ct + \dfrac{1}{c} \sin x \sin ct$

2. Solve $u_{tt} = c^2 u_{xx},\ u(x,0) = \log(1+x^2),\ u_t(x,0) = 4 + x$.

$$u(x,t) = \frac{1}{2}\left\{\ln\left[1+(x+ct)^2\right] + \ln\left[1+(x-ct)^2\right]\right\} + 5t$$

3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

$$\left(\frac{l}{4} - a\right)\sqrt{\frac{\rho}{T}}$$

4. Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x+ct) + g(x-ct)$.

7. If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,\ u(x,0) = x^2,\ u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

By $(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$， we know that u (x,t)=f(4x+t)+g(x-t) is the general solution。

(It is easy to check that both u(x,t)=f(4x+t) and u(x,t)=g(x-t) are solutions of the PDE。

Let $v = u_x + u_t$ then $v_x - 4v_t = 0 \Rightarrow v(x,t) = \phi(4x+t)$

$u_x + u_t = \phi(4x+t)$

The honogeneous solution is u(x,t)=g(x-t)

Let u(x,t)=f(4x+t) be the particular solutin，then

$u_x + u_t = 4f'(\xi) + f'(\xi) = \phi(\xi), \xi = 4x + 5$ ，$f'(\xi) = \frac{1}{5}\phi(\xi)$

That is u (x,t)=f(4x+t)+g(x-t) with $f'(\xi) = \frac{1}{5}\phi(\xi)$)

$f(4x) + g(x) = x^2 \Rightarrow 4f'(4x) + g'(x) = 2x$

$u_t(x,0) = f'(4x) - g'(x) = e^x$

$$\begin{cases} f'(4x) = \frac{1}{5}(e^x + 2x) \\ g'(x) = \frac{1}{5}(2x - 4e^x) \end{cases}$$

Integrating both sides，we have

$$\begin{cases} \frac{1}{4}f(4x) = \frac{1}{5}e^x + \frac{1}{5}x^2 + A' \\ g(x) = -\frac{4}{5}e^x + \frac{1}{5}x^2 + B \end{cases}$$

$f(4x) = \frac{4}{5}e^x + \frac{4}{5}x^2 + A, A = 4A'$ ，with A+B=0

$$f(4x) = \frac{4}{5}e^x + \frac{4}{5}x^2 + A, A = 4A' \Rightarrow f(x) = \frac{4}{5}e^{\frac{x}{4}} + \frac{4}{5}\left(\frac{x}{4}\right)^2 + A$$

Then $u(x,t) = f(4x+t) + g(x-t) = \frac{4}{5}\left(e^{x+\frac{t}{4}} - e^{x-t}\right) + x^2 + \frac{1}{4}t^2$

10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$.

11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2,\ x + a/2) \cap (-a,\ a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.

8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{(“spherical wave equation”).}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
(b) Solve for v using (3) and thereby solve the spherical wave equation.
(c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

Section 2.1

1. $e^x \cosh ct + (1/c) \sin x \sin ct$
3. $(l/4 - a)(\sqrt{\rho/T})$
4. As in the text, $u_t + cu_x = h(x + ct)$. Let $w = u - f(x - ct)$. Show that $w_t + cw_x = 0$ and then find the form of w.

6. Let $m(t) = \max_x u(x, t)$. Then $m(t) = t$ for $0 \le t \le a/c$, and $m(t) = a/c$ for $t \ge a/c$.
8. (b) $u(r, t) = (1/r)\,[f(r + ct) + g(r - ct)]$
(c) $(1/2r)\{(r + ct)\phi(r + ct) + (r - ct)\phi(r - ct)\} +$
$(1/2cr)\int_{r-ct}^{r+ct} s\psi(s)\,ds$
9. $\frac{4}{5}(e^{x+t/4} - e^{x-t}) + x^2 + \frac{1}{4}t^2$
11. $u = f(3x - t) + g(x - 3t) - \frac{1}{16}\sin(x + t)$

§ 2.2 Causality(因果關係) and Energy

$\rho u_{tt} = T u_{xx}$

$KE = \frac{1}{2}\rho \int u_t^2 dx$

$$\frac{dKE}{dt} = \rho \int u_t u_{tt} dx = T \int u_t u_{xx} dx = T \int u_t d(u_x) = T u_t u_x \Big|_{-\infty}^{\infty} - T \int u_{tx} u_x dx$$

$$= -\frac{d}{dt} \int \frac{1}{2} T u_x^2 dx$$

Let $PE = \frac{1}{2} T \int u_x^2 dx$,and E=KE+PE

then $\frac{dE}{dt} = 0$, $E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx$ is independent of t。

Exercises

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)

2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.

3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.

4. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$

for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

5. For the *damped* string, equation (1.3.3), show that the energy decreases.

6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right),$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)