Introduction to Partial Differential Equations Peter J. Olver

Exercises

1.1. Classify each of the following differential equations as ordinary or partial, and equilibrium or dynamic; then write down its order. (a) $\dfrac{du}{dx} + x\,u = 1$, (b) $\dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u}{\partial x} = x$,
(c) $u_{tt} = 9\,u_{xx}$, (d) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}$, (e) $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = x^2 + y^2$,
(f) $\dfrac{d^2 u}{dt^2} + 3\,u = \sin t$, (g) $u_{xx} + u_{yy} + u_{zz} + (x^2 + y^2 + z^2)u = 0$, (h) $u_{xx} = x + u^2$,
(i) $\dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} + u\,\dfrac{\partial u}{\partial x} = 0$, (j) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y\,\partial z} = u$, (k) $u_{tt} = u_{xxxx} + 2\,u_{xxyy} + u_{yyyy}$.

1.2. In two space dimensions, the *Laplacian* is defined as the second-order partial differential operator $\Delta = \partial_x^2 + \partial_y^2$. Write out the following partial differential equations in (*i*) Leibniz notation; (*ii*) subscript notation: (a) the Laplace equation $\Delta u = 0$; (b) the Poisson equation $-\Delta u = f$; (c) the two-dimensional heat equation $\partial_t u = \Delta u$; (d) the von Karman plate equation $\Delta^2 u = 0$.

1.3. Answer Exercise 1.2 for the three-dimensional Laplacian $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

1.4. Identify the independent variables, the dependent variables, and the order of the following systems of partial differential equations: (a) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$;
(b) $u_{xx} + v_{yy} = \cos(x + y)$, $u_x v_y - u_y v_x = 1$; (c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial x}$, $\dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$;
(d) $u_t + u\,u_x + v\,u_y = p_x$, $v_t + u\,v_x + v\,v_y = p_y$, $u_x + v_y = 0$;
(e) $u_t = v_{xxx} + v(1 - v)$, $v_t = u_{xxy} + v\,w$, $w_t = u_x + v_y$.

§ Initial conditions and boundary conditions

1.5. Show that the following functions $u(x, y)$ define classical solutions to the two-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Be careful to specify an appropriate domain.
(a) $e^x \cos y$, (b) $1 + x^2 - y^2$, (c) $x^3 - 3\,x\,y^2$, (d) $\log(x^2 + y^2)$, (e) $\tan^{-1}(y/x)$, (f) $\dfrac{x}{x^2 + y^2}$.

(c) $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1 + x^2}$

1.6. Find all solutions $u = f(r)$ of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ that depend only on the radial coordinate $r = \sqrt{x^2 + y^2}$.

1.7. Find all (real) solutions to the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ of the form $u = \log p(x, y)$, where $p(x, y)$ is a quadratic polynomial.

1.8. (a) Find all quadratic polynomial solutions of the three-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$ (b) Find all the homogeneous cubic polynomial solutions.

1.9. Find all polynomial solutions $p(t,x)$ of the heat equation $u_t = u_{xx}$ with $\deg p \le 3$.

1.10. Show that each of the following functions $u(t,x)$ is a solution to the wave equation $u_{tt} = 4\,u_{xx}$: (a) $4t^2 - x^2$; (b) $\cos(x+2t)$; (c) $\sin 2t \cos x$; (d) $e^{-(x-2t)^2}$.

1.11. Find all polynomial solutions $p(t,x)$ of the wave equation $u_{tt} = u_{xx}$ with
(a) $\deg p \le 2$, (b) $\deg p = 3$.

1.12. Suppose $u(t,x)$ and $v(t,x)$ are C^2 functions defined on $\mathbb{R}^2$ that satisfy the first-order system of partial differential equations $u_t = v_x,\ \ v_t = u_x$.
(a) Show that both u and v are classical solutions to the wave equation $u_{tt} = u_{xx}$. Which result from multivariable calculus do you need to justify the conclusion?
(b) Conversely, given a classical solution $u(t,x)$ to the wave equation, can you construct a function $v(t,x)$ such that $u(t,x), v(t,x)$ form a solution to the first-order system?

1.13. Find all solutions $u = f(r)$ of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ that depend only on the radial coordinate $r = \sqrt{x^2+y^2+z^2}$.

1.14. Let $u(x,y)$ be defined on a domain $D \subset \mathbb{R}^2$. Suppose you know that all its second-order partial derivatives, $u_{xx}, u_{xy}, u_{yx}, u_{yy}$, are defined and continuous on all of D. Can you conclude that $u \in \mathrm{C}^2(D)$?

1.15. Write down a partial differential equation that has
(a) no real solutions; (b) exactly one real solution; (c) exactly two real solutions.

1.16. Let $u(x,y) = xy\,\dfrac{x^2-y^2}{x^2+y^2}$ for $(x,y) \ne (0,0)$, while $u(0,0) = 0$. Prove that

$$\frac{\partial^2 u}{\partial x\,\partial y}(0,0) = 1 \ne -1 = \frac{\partial^2 u}{\partial x\,\partial y}(0,0).$$

Explain why this example does not contradict the theorem on the equality of mixed partials.

§ Linear and nonlinear equations

1.17. Classify the following differential equations as either
(*i*) homogeneous linear; (*ii*) inhomogeneous linear; or (*iii*) nonlinear:
(a) $u_t = x^2 u_{xx} + 2x\,u_x$, (b) $-u_{xx} - u_{yy} = \sin u$; (c) $u_{xx} + 2y\,u_{yy} = 3$;
(d) $u_t + u\,u_x = 3u$; (e) $e^y u_x = e^x u_y$; (f) $u_t = 5u_{xxx} + x^2 u + x$.

1.18. Write down all possible solutions to the Laplace equation you can construct from the various solutions provided in Exercise 1.5 using linear superposition.

1.19. (a) Show that the following functions are solutions to the wave equation $u_{tt} = 4u_{xx}$:
(*i*) $\cos(x - 2t)$, (*ii*) e^{x+2t}; (*iii*) $x^2 + 2xt + 4t^2$.
(b) Write down at least four other solutions to the wave equation.

1.20. The displacement $u(t,x)$ of a forced violin string is modeled by the partial differential equation $u_{tt} = 4u_{xx} + F(t,x)$. When the string is subjected to the external forcing $F(t,x) = \cos x$, the solution is $u(t,x) = \cos(x-2t) + \frac{1}{4}\cos x$, while when $F(t,x) = \sin x$, the solution is $u(t,x) = \sin(x-2t) + \frac{1}{4}\sin x$. Find a solution when the forcing function $F(t,x)$ is
(a) $\cos x - 5\sin x$, (b) $\sin(x-3)$.

(a) $u(t,x) = \cos(x-2t) + \sin(x-2t) + \dfrac{1}{4}\cos x - \dfrac{5}{4}\sin x$

1.21. (a) Show that the partial derivatives $\partial_x[f] = \dfrac{\partial f}{\partial x}$ and $\partial_y[f] = \dfrac{\partial f}{\partial y}$ both define linear operators on the space of continuously differentiable functions $f(x,y)$. (b) For which values of a, b, c, d is the differential operator $L[f] = a\,\dfrac{\partial f}{\partial x} + b\,\dfrac{\partial f}{\partial y} + c\,f + d$ linear?

1.22. (a) Prove that the Laplacian $\Delta = \partial_x^2 + \partial_y^2$ defines a linear differential operator.
(b) Write out the Laplace equation $\Delta[u] = 0$ and the Poisson equation $-\Delta[u] = f$.

1.23. Prove that, on $\mathbb{R}^3$, the gradient, curl, and divergence all define linear operators.

1.24. Let L and M be linear partial differential operators. Prove that the following are also linear partial differential operators: (a) $L - M$, (b) $3L$, (c) $f\,L$, where f is an arbitrary function of the independent variables; (d) $L \circ M$.

1.25. Suppose L and M are linear differential operators and let $N = L + M$.
(a) Prove that N is a linear operator. (b) *True or false*: If u solves $L[u] = f$ and v solves $M[v] = g$, then $w = u + v$ solves $N[w] = f + g$.

Theorem 1.7. *Let $v_1, \ldots, v_k$ be solutions to the inhomogeneous linear systems $L[v_1] = f_1, \ \ldots \ , L[v_k] = f_k$, involving the same linear operator L. Then, given any constants $c_1, \ldots, c_k$, the linear combination $v = c_1 v_1 + \cdots + c_k v_k$ solves the inhomogeneous system $L[v] = f$ for the combined forcing function $f = c_1 f_1 + \cdots + c_k f_k$.*

1.27. Solve the following inhomogeneous linear ordinary differential equations:
(a) $u' - 4u = x - 3$, (b) $5u'' - 4u' + 4u = e^x \cos x$, (c) $u'' - 3u' = e^{3x}$.

(a) $u(x) = ce^{4x} - \dfrac{1}{4}x + \dfrac{11}{16}$

(b) $5x^2 - 4x + 4 = 0, x = \dfrac{2 \pm 4i}{5}$，所以 齊次解為 $u(x) = c_1 e^{\frac{2x}{5}} \cos\dfrac{4x}{5} + c_2 e^{\frac{2x}{5}} \sin\dfrac{4x}{5}$

特別解 取 $u(x) = ae^x \sin x$ 代入， 可得 $a = \dfrac{1}{6}$

(c) 齊次解 $u(x) = c_1 e^{3x} + c_2$

特別解 $u(x) = \dfrac{1}{3} x e^{3x}$

1.28. Use superposition to solve the following inhomogeneous ordinary differential equations:
(a) $u' + 2u = 1 + \cos x$, (b) $u'' - 9u = x + \sin x$, (c) $9u'' - 18u' + 10u = 1 + e^x \cos x$,
(d) $u'' + u' - 2u = \sinh x$, where $\sinh x = \frac{1}{2}(e^x - e^{-x})$, (e) $u''' + 9u' = 1 + e^{3x}$.

§

1. Verify that $u(x,y) = x^2 + y^2$ is a solution of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} - 2u = 0$

2. Verify that $u(x,y) = e^{-2y} \cos x$ is a solution of $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} - u = 0$

3. Find the general solution of $u_{xy} = \sin x + y$

x 視為常數，兩邊對 y 積分 得 $u_x(x,y) = y \sin x + \dfrac{1}{2} y^2 + h(x)$

y 視為常數，兩邊對 x 積分 得 $u(x,y) = -y \cos x + \dfrac{1}{2} x y^2 + \int h(x)dx + g(y)$

其中 $\int h(x)dx$ 寫成 f(x)

$u(x,y) = -y\cos x + \frac{xy^2}{2} + f(x) + g(y)$

4. Find the general solution of $x^2 u_{xy} + xu_y = y, x > 0$

先寫成$u_{xy} + \frac{1}{x}u_y = \frac{1}{x^2}y$

對 y 積分，$u_x + \frac{1}{x}u = \frac{1}{2}\frac{1}{x^2}y^2 + h(x)$

$xu_x + u = \frac{1}{2x}y^2 + xh(x)$ ，$xu_x + u = \frac{\partial}{\partial x}(xu)$

對 x 積分，$xu = \frac{y^2 \ln x}{2} + \int xh(x)dx + f(y)$ let $\int xh(x)dx = g(x)$

$u(x,y) = \frac{y^2 \ln x}{2x} + \frac{1}{x}f(y) + g(x)$

5. Find the general solution of $u_{yy} - x^2 u = x\sin y$ 一般解=齊次解＋特別解

齊次解$u_{yy} = x^2 u \Rightarrow u(x,y) = f(x)e^{-xy} + g(x)e^{xy}$

(與解 $y'' - ky = 0$ 一樣 now $k = x^2$ $\lambda = \pm x$ $y = c_1 e^{-xt} + c_2 e^{xt}$)

設(特別解) $u_p = A(x)\sin y + B(x)\cos y$ 代入原方程式得

A=$\frac{-x}{1+x^2}, B = 0$ 所以$u(x,y) = f(x)e^{-xy} + g(x)e^{xy} - \frac{x\sin y}{1+x}$

6. Find the general solution of $u_{xy} = 2u_x + e^{x+y}$

Let $v(x,y) = u_x(x,y)$ ，then $v_y - 2v = e^{x+y}$ ，視 x 為常數 解 ODE

積分因子為$e^{\int -2dy} = e^{-2y}$ ，$e^{-2y}(v_y - 2v) = e^{x-y}$ ，$e^{-2y}(v_y - 2v) = \frac{d}{dy}(e^{-2y} \cdot v)$

$e^{-2y} \cdot v = -e^{x-y} + f(x)$ ，$u_x = v = -e^{x+y} + e^{2y} f(x)$

$u(x,y) = -e^{x+y} + e^{2y}\int f(x)dx + g(y)$

7. 解$u_x + yu = 2xy$

$$u(x,y) = f(y)e^{-xy} + 2x - \frac{2}{y}$$

8. Verify that u(x,y)=f(x-2y)+g(x+y) is the general solution of $2u_{xx} - u_{xy} - u_{yy} = 0$