Lecture 1 The General Theory for One First-Order Equations
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以下是 Frobenius 定理：

1. $\varphi: M \to N$ 是一個可微映射且 $d\varphi_p: T_pM \to T_{\varphi(p)}N$ 是 1-1(injective) for $\forall p \in M$

則稱 φ 是一個浸射(immersion)。

2. 對 $\forall p \in M$ ，D_p 是切平面 T_pM 中的 k-dim 線性子空間。

若 $\forall p_0 \in M$ ，存在 $C^\infty - immersion \quad \varphi: U \to M$ ，

使得 $p_0 \in \varphi(U)$ ，且 $T_{\varphi(x)}(\varphi(U)) = D_{\varphi(x)}$ for $\forall x \in U$ ，則稱 D 為可積分

白話文是這麼說的：(大域微分幾何 黃武雄 p.129)
設 $M = R^3$ ，在 R^3 中每一點先可微地指定一平面，得到一個 2 維平面場 D。

設通過每一點 p_0 有一個曲面 $\alpha(U)$ ，使得在 $\alpha(U)$ 上任一點的切平面都是原來指定的平面，那麼 我們就稱 D 為可積的(integrable)。

In the case of an ODE，a locally integrable vector field (one having integral curves) is defined on a manifold。
For a PDE，a subspace of the tangent space of dimension greater than 1 is defined at each point of the manifold。
As is known，even a field of two-dimensional planes in three-dimensional space is in general not integrable。
這是在說 Frobenius 定理。

Example. In a space with coordinates x, y, and z we consider the field of planes given by the equation $dz = y\,dx$. (This gives a linear equation for the coordinates of the tangent vector at each point, and that equation determines a plane.)

Problem 1. Draw this field of planes and prove that it has no integral surface, that is, no surface whose tangent plane at every point coincides with the plane of the field.

Let $\omega = dz - ydx$，then $d\omega = -dy \wedge dx = dx \wedge dy$ ，$d\omega \wedge \omega = dx \wedge dy \wedge dz \neq 0$
ω沒有積分因子，所以是不可積。
也就是沒有 integral surface(submanifold of R^3)
If so，how to draw this field of planes？

1. 有積分曲面的例子：

$\omega = yzdx + xzdy + z^2dz$

$$d\omega = d(yz)\wedge dx + d(xz)\wedge dy + d(z^2)\wedge dz$$
$$= (zdy + ydz)\wedge dx + (zdx + xdz)\wedge dy$$
$$= ydz\wedge dx + xdz\wedge dy$$

$d\omega\wedge\omega = xyzdz\wedge dx\wedge dy + xyzdz\wedge dy\wedge dx = 0$
所以$\omega = 0$有積分曲面。

$\omega = yzdx + xzdy + z^2dz = z(ydx + xdy + zdz)$

取 f=z，$g = xy + \frac{1}{2}z^2$ 則$\omega = fdg$

積分曲面是$xy + \frac{1}{2}z^2 = constant$ 。

2. 沒有積分曲面的例子
$\omega = dz - ydx - dy$
On the plane x=at, y=bt，the equation $\omega = 0$ becomes dz=(abt+b)dt
$z = \frac{1}{2}abt^2 + bt + c$ and we arrive at the surface $z = \frac{1}{2}xy + y + c$

But on the parabolic cylinder x=at，y=bt^2 we have $dz = (abt^2 + 2bt)dt$

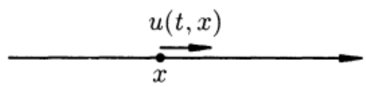
$z = \frac{1}{3}abt^3 + bt^2 + c$ ，$z = \frac{1}{3}xy + y + c$ a different family of surfaces。
The reason for this failure to obtain integral surfaces is seen from
$d\omega = -dy\wedge dx, d\omega\wedge\omega = -dy\wedge dx\wedge dz \neq 0$

§ The general first-order PDE has the form：

$F(x_1, x_2, \ldots, x_n, y, p_1, p_2, \ldots, p_n) = 0$ ，where $p_i = \frac{\partial u}{\partial x_i} = u_{x_i}$

$u_t + uu_x = 0$ 稱 Eikonal equation in geometric optics(幾何光學) 程函方程是波傳播問題中遇到的非線性 PDE。

A particle on a line



Consider the field u(t,x) of velocities of particles moving freely along a line。

The law of free motion of a particle has the form $x = \varphi(t) = x_0 + vt$ ，where v is the velocity of the particle。

The function φ satifies Newton's equation $\frac{d^2\varphi}{dt^2} = 0$。

We now give a description of the motion in term of the velocity field u：

By definition $\frac{d\varphi}{dt} = u(t, \varphi(t))$

$\varphi(t)$ is the flow of the vector field u。

Differentiate with respect to t，we obtain the Euler equation：

$$\frac{d^2\varphi}{dt^2} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} = u_t + u_x u = 0$$

§ 從牛頓方程式 $\frac{d^2\varphi}{dt^2} = 0$ 可推出 尤拉方程式 $u_t + uu_x = 0$

反之，可從尤拉方程式推出牛頓方程式。

即 這些運動的描述，用一個向量場的尤拉方程式與用粒子的牛頓方程式是等價的。

v=v(x)是一 manifold 的向量場，考慮此方程式 $L_v(u) = 0$，L_v是一個 operator

表示在向量場方向的導數(Lie derivative)

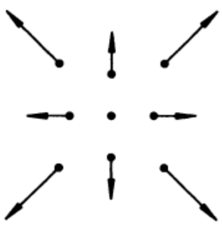
用座標表示為 $v_1\frac{\partial u}{\partial x_1} + v_2\frac{\partial u}{\partial x_2} + \ldots + v_n\frac{\partial u}{\partial x_n} = 0$

是一個齊次線性一階 PDE。

函數 u 是此 PDE 的解的充要條件是 u 沿著向量場 v 的 phase curves 是常數。

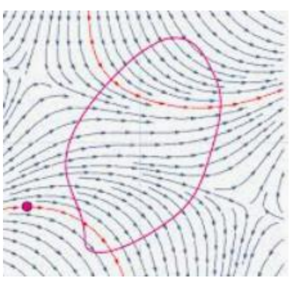
因此這個解稱為此向量場的 first integrals。

An Eulerian field



左圖向量場 $v = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$

解 $L_v(u) = 0$ for this field



向量場 $A = A_x i + A_y j + A_z k$

P(x,y,z)為向量線上任一點，其向徑為

r=xi+yj+zk，則 dr=dxi+dyj+dzk 在 P 點與向量線相切的向量共線。因此

$\frac{dx}{A_x} = \frac{dy}{A_y} = \frac{dz}{A_z}$ 這就是向量線所應滿足的微分方程。

例 靜電場 $E=\frac{q}{4\pi\epsilon r^3}r$

$E=\frac{q}{4\pi\epsilon r^3}(xi+yi+zk)$ $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ 解得 $\begin{cases} y=c_1x \\ z=c_2y \end{cases}$

其向量線就是上面的 Euler field。

例 求向量場$A=xzi+yzj-(x^2+y^2)k$ 在 P(2,-1,1)的向量線方程

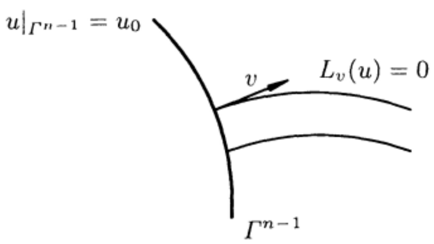
$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}$

解得 $\begin{cases} y=c_1x \\ x^2+y^2+z^2=c_2 \end{cases}$ 是以原點為中心的同心圓族

P 點代入 得

$\begin{cases} y=-\frac{1}{2}x \\ x^2+y^2+z^2=6 \end{cases}$

§ The Cauchy Problem



Consider a smooth hypersurface Γ^{n-1} in x-space。
假設 M 是 n 維流形，則 M 的任意 n-1 維子流形即為超曲面。
The Cauchy problem：
Find a solution of the equation $L_v(u)=0$ that coinsides with a given function on this hypersurface。

A point of the hypersurface is called noncharacteristic if the field v is transversal (橫向的 n.截線)to the surface at that point。
A Cauchy problem can be an initial value problem or a boundary value problem。

[Cauchy problem for First Order PDE]

In coordinates a first-order quasilinear equation has the form

$$a_1(x,u)\frac{\partial u}{\partial x_1}+\cdots+a_n(x,u)\frac{\partial u}{\partial x_n}=f(x,u)\,. \qquad (*)$$

The vector field v (in the space of independent variables) is called the *characteristic field* of the linear equation $L_v(u)=f$.

Example

$2u_x - 3u_y + 2u = 2x$ such that $u = x^2 \; for \; y = -\dfrac{x}{2}$

解 1.

Let $u = Ae^{\alpha x + \beta y}$ to get the general solution of the solution of homogeneous equation $2u_x - 3u_y + 2u = 0$, $u(x,y) = e^{-x} f(3x+2y)$

c.f. PDE1042OCC

And u=x-1 is a particular solution。

So，the solution of the PDE is $u(x,y) = (x-1) + e^{-x} f(3x+2y)$

Then we apply the initial condition in the general solution where f is an arditrary function。

$$u = x^2 \; for \; y = -\frac{x}{2}$$

$$x^2 = x - 1 + e^{-x} f(2x)$$
$$f(2x) = e^x \left(x^2 - x + 1\right)$$
$$f(x) = e^{\frac{x}{2}} \left(\frac{x^2}{4} - \frac{x}{2} + 1\right)$$
$$u(x,y) = x - 1 + e^{-x} \times 2^{\frac{3x+2y}{2}} \left[\frac{(3x+y)^2}{4} - \frac{3x+2y}{2} + 1\right]$$
$$= x - 1 + \left[\frac{(3x+2y)^2}{4} - \frac{3x+2y}{2} + 1\right] \times e^{\frac{x+2y}{2}}$$

The solution is unique。

解 2.

By Lagrange method

$$\frac{dx}{2} = \frac{dy}{-3} = \frac{du}{2x-2u}$$

$$\frac{dx}{2} = \frac{dy}{-3} \Rightarrow 3x + 2y = c_1$$

$$\frac{dx}{2} = \frac{du}{2x-2u} \Rightarrow (x-u)dx - du = 0$$

The integrating factor is $\mu = \exp\left(\int dx\right) = e^x$，then

$$\varphi(x,u) = -e^x u + xe^x - e^x = e^x(x-1) - e^x u = c_2$$

The general solution is $\phi(c_1, c_2) = 0$

As $u = x^2 \; for \; y = -\dfrac{x}{2}$，$c_1 = 2x, c_2 = e^x\left(-x^2 + x - 1\right) = e^{\frac{c_1}{2}}\left(-\left(\dfrac{c_1}{2}\right)^2 + \dfrac{c_1}{2} - 1\right)$

In general case

$$c_1 = 3x + 2y, c_2 = e^{\frac{3x+2y}{2}}\left[-\left(\frac{3x+2y}{2}\right)^2 + \frac{3x+2y}{2} - 1\right]$$

$u(x,y) = x - 1 - e^{-x}c_2$ 上式的c_2 代入 得

$$u(x,y) = x - 1 + \left[\frac{(3x+2y)^2}{4} - \frac{3x+2y}{2} + 1\right] e^{\frac{x+2y}{2}}$$

z=u(x,y) is the hypersurface of R^3，satisfying the PDE with the boundary condition (the Cauchy problem)。

Theorem 2. *The Cauchy problem has a unique solution in a neighborhood of each noncharacteristic point.*

[PDE and Characteristics]

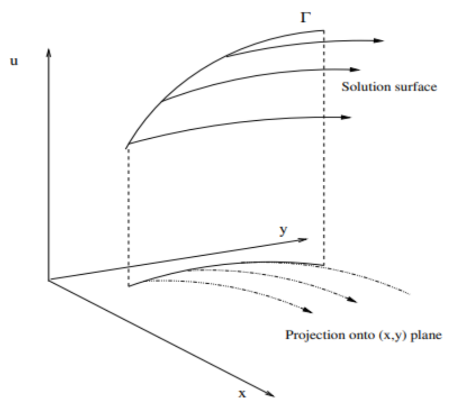


Figure 1: The solution surface and the characteristic projection.

Consider the following first order partial differential equation for the dependent variables $u(x, y)$

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u). \tag{1}$$

This is a *quasi-linear* partial differential equation, because it is linear in the derivatives of $u(x, y)$. The equation is to be integrated subject to *Cauchy data*: $u(x, y)$ is given on a curve Γ. In parametric form, this corresponds to

$$u = u_0(\xi) \qquad \text{on} \qquad x = x_0(\xi),\ y = y_0(\xi), \tag{2}$$

where the parameter $\xi_1 < \xi < \xi_2$. Here u_0, x_0 and y_0 are smooth functions of ξ and there is no value of ξ for which $\mathrm{d}x_0/\mathrm{d}\xi = \mathrm{d}y_0/\mathrm{d}\xi = 0$.

We write $\frac{\mathrm{d}x}{\mathrm{d}s} = a$ and $\frac{\mathrm{d}y}{\mathrm{d}s} = b$ for some parameter s and then the pde (1) and Cauchy data (2) are given by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a, \ \frac{\mathrm{d}y}{\mathrm{d}s} = b \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = c, \tag{5}$$

subject to $x = x_0(\xi)$, $y = y_0(\xi)$, $u = u_0(\xi)$ for $\xi_1 < \xi < \xi_2$ on $s = 0$. The projection of the solution $u(x, y)$ onto the (x, y)-plane is termed the *characteristic projection* and the curves $\frac{\mathrm{d}x}{\mathrm{d}s} = a$ and $\frac{\mathrm{d}y}{\mathrm{d}s} = b$ are the *characteristics*. (See figure 1 for a sketch of the characteristics.)

Examples

1. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, with\, u(x,0) = f(x)$

2. $\frac{\partial u}{\partial t} + e^x\frac{\partial u}{\partial x} = 0, with\, u(x,0) = \cosh(x)$

$\frac{dt}{1} = \frac{dx}{e^x} = \frac{du}{0}$ 換個方式寫 $\frac{dt}{ds} = 1, \frac{dx}{ds} = e^x, \frac{du}{ds} = 0$

$t = 0, x = \xi, u = \cosh\xi\, at\, s = 0$

s=t，$\frac{ds}{dx} = e^{-x} \Rightarrow s = -e^{-x} + c$ 因為 s=0，$x = \xi \Rightarrow c = e^{-\xi}$

消掉參數 s，$e^{-\xi} = t + e^{-x} \Rightarrow \xi = -\ln\left(t + e^{-x}\right)$

$u(x,t) = \cosh\xi = \frac{e^{\xi} + e^{-\xi}}{2} = \frac{1}{2}\left(t + e^{-x}\right) + \frac{1}{2}\left(t + e^{-x}\right)^{-1}$

3. $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = ye^x\, with\, u(x,x) = \sin x$

Example

For the Euler equation $u_t + uu_x = 0$ the equation of the characteristic is equivalent to Newton's equation $\dot{t} = 1, \dot{x} = u, \dot{u} = 0$

即 $\frac{dt}{ds} = 1, \frac{dx}{ds} = u, \frac{du}{ds} = 0$