§ What are Partial Differential Equations？

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For example, the three-dimensional *Navier–Stokes equations*

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} &= -\,\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} &= -\,\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} &= -\,\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right),\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned} \tag{1.4}$$

Is a second-order system of differential equations，while $\nu \geq 0$ is a fixed constant。

v=0 is known as Euler equations。

The Navier-Stokes equation are fundamental in fluid mechanics。

Unsolved problem in Clay Mathematics Institute。

Example

$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ is the heat equation。

1. $u(t,x) = t + \frac{1}{2}x^2$ ， $D = R^2$

2. $u(t,x) = \dfrac{e^{\frac{-x^2}{4t}}}{2\sqrt{\pi t}}$ ，D={t>0}

3. $u(t,x) = e^{-t}(\cos x + i\sin x)$

Are solutions of the equation。

Incidentally, most partial differential equations arising in physical applications are real, and, although complex solutions often facilitate their analysis, at the end of the day we require real, physically meaningful solutions. A notable exception is quantum mechanics, which is an inherently complex-valued physical theory. For example, the one-dimensional *Schrödinger equation*

$$\mathrm{i}\,\hbar\,\frac{\partial u}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\frac{\partial^2 u}{\partial x^2} + V(x)\,u, \tag{1.9}$$

with $\hbar$ denoting *Planck's constant*, which is real, governs the dynamical evolution of the complex-valued wave function $u(t,x)$ describing the probabilistic distribution of a quantum particle of mass m, e.g., an electron, moving in the force field prescribed by the (real) potential function $V(x)$. While the solution u is complex-valued, the independent variables t, x, representing time and space, remain real.

Exerise

1.1-1.4

§ Initial conditions and boundary conditions

There are three principal types of boundary value problems that arise in most applications。

1. Dirichlet boundary condition
2. Neumann boundary condition
3. Mixed boundary value problem

Exercise

1.5-1.16

§ Linear and nonlinear equations

Homogeneous linear equation

Examples

1. $\dfrac{d^2u}{dx^2} + \dfrac{u}{1+x^2} = 0$
2. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - u$
3. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$
4. $\dfrac{\partial u}{\partial t} = e^x \dfrac{\partial^2 u}{\partial x^2} + \cos(x-t)u$

On the other hand，Burgers' equation

$\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}$ is not linear，since the second involves the product of u and u_x 。

Theorem 1.4. *If* $u_1, \ldots, u_k$ *are solutions to a common homogeneous linear equation* $L[u] = 0$, *then the* linear combination, *or* superposition, $u = c_1 u_1 + \cdots + c_k u_k$ *is a solution for any choice of constants* $c_1, \ldots, c_k$.

Theorem 1.6. *Let* $v_\star$ *be a particular solution to the inhomogeneous linear equation* $L[v_\star] = f$. *Then the general solution to* $L[v] = f$ *is given by* $v = v_\star + u$, *where* u *is the general solution to the corresponding homogeneous equation* $L[u] = 0$.

Theorem 1.7. *Let* $v_1, \ldots, v_k$ *be solutions to the inhomogeneous linear systems* $L[v_1] = f_1, \ \ldots \ , L[v_k] = f_k$, *involving the same linear operator* L. *Then, given any constants* $c_1, \ldots, c_k$, *the* linear combination $v = c_1 v_1 + \cdots + c_k v_k$ *solves the inhomogeneous system* $L[v] = f$ *for the combined forcing function* $f = c_1 f_1 + \cdots + c_k f_k$.

Exercise

1.17-1.28