§ Wave in space

9.1 Energy and causality

The characteristic cone

Conservation of energy

Principle of causality

Exercises

1. Find all the three-dimensional plane waves; that is, all the solutions of the wave equation of the form $u(\mathbf{x}, t) = f(\mathbf{k} \cdot \mathbf{x} - ct)$, where $\mathbf{k}$ is a fixed vector and f is a function of one variable.

2. Verify that $(c^2t^2 - x^2 - y^2 - z^2)^{-1}$ satisfies the wave equation except on the light cone.

3. Verify that $(c^2t^2 - x^2 - y^2)^{-1/2}$ satisfies the two-dimensional wave equation except on the cone $\{x^2 + y^2 = c^2t^2\}$.

4. *(Lorentz invariance of the wave equation)* Thinking of the coordinates of space-time as 4-vectors (x, y, z, t), let Γ be the diagonal matrix with the diagonal entries 1, 1, 1, −1. Another matrix L is called a *Lorentz transformation* if L has an inverse and $L^{-1} = \Gamma\, {}^tL\Gamma$, where ${}^tL$ is the transpose.
   (a) If L and M are Lorentz, show that LM and L^{-1} also are.
   (b) Show that L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$, where $m(\mathbf{v}) = x^2 + y^2 + z^2 - t^2$ is called the *Lorentz metric*.
   (c) If $u(x, y, z, t)$ is any function and L is Lorentz, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that
   $$U_{xx} + U_{yy} + U_{zz} - U_{tt} = u_{xx} + u_{yy} + u_{zz} - u_{tt}.$$
   (d) Explain the meaning of a Lorentz transformation in more geometrical terms. (*Hint:* Consider the level sets of $m(\mathbf{v})$.)

5. Prove the principle of causality in two dimensions.

6. (a) Derive the conservation of energy for the wave equation in a domain D with homogeneous Dirichlet or Neumann boundary conditions.
   (b) What about the Robin condition?

7. For the boundary condition $\partial u/\partial n + b\, \partial u/\partial t = 0$ with $b > 0$, show that the energy defined by (6) decreases.

8. Consider the equation $u_{tt} - c^2\Delta u + m^2 u = 0$, where $m > 0$, known as the *Klein–Gordon equation*.
   (a) What is the energy? Show that it is a constant.
   (b) Prove the causality principle for it.

9.2 The wave equation in space-time

Solution in two dimensions

Exercises

1. Prove that $\Delta(\overline{u}) = (\overline{\Delta u})$ for any function; that is, the laplacian of the average is the average of the laplacian. (*Hint:* Write Δu in spherical coordinates and show that the angular terms have zero average on spheres centered at the origin.)

2. Verify that (3) is correct in the case of the example $u(x, y, z, t) \equiv t$.

3. Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = y$, by use of (3).

4. Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = x^2 + y^2 + z^2$. (*Hint:* Use (5).)

5. Where does a three-dimensional wave have to vanish if its initial data ϕ and ψ vanish outside a sphere?

7. (a) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, and $\psi|\mathbf{x}| \equiv 0$, where A is a constant. (This is somewhat like the plucked string.) (*Hint:* Differentiate the solution in Exercise 6(b).)

   (b) Sketch the regions in space-time that illustrate your answer. Where does the solution have jump discontinuities?
   (c) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \to \infty.$$

8. Carry out the derivation of the second term in (3).

6. (a) Let S be the sphere of center $\mathbf{x}$ and radius R. What is the surface area of $S \cap \{|\mathbf{x}| < \rho\}$, the portion of S that lies within the sphere of center $\mathbf{0}$ and radius ρ?
(b) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Sketch the regions in space-time that illustrate your answer. (This is like the hammer blow of Section 2.1.)
(c) Sketch the graph of the solution (u versus $|\mathbf{x}|$) for $t = \frac{1}{2}$, 1, and 2, assuming that $\rho = c = A = 1$. (This is a "movie" of the solution.)
(d) Sketch the graph of u versus t for $|\mathbf{x}| = \frac{1}{2}$ and 2, assuming that $\rho = c = A = 1$. (This is what a stationary observer sees.)
(e) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \to \infty.$$

(*Hint:* (a) Divide into cases depending on whether one sphere contains the other or not. Use the law of cosines. (b) Use Kirchhoff's formula.)

9. (a) For any solution of the three-dimensional wave equation with initial data vanishing outside some sphere, show that $u(x, y, z, t) = 0$ for fixed (x, y, z) and large enough t.
(b) Prove that $u(x, y, z, t) = O(t^{-1})$ *uniformly* as $t \to \infty$; that is, prove that $t \cdot u(x, y, z, t)$ is a bounded function of x, y, z, and t. (*Hint:* Use Kirchhoff's formula.)

10. Derive the mean value property of harmonic functions $u(x, y, z)$ by the following method. A harmonic function is a wave that happens not to depend on time, so that its mean value $\overline{u}(r, t) = \overline{u}(r)$ satisfies (5). Deduce that $\overline{u}(r) = u(\mathbf{0})$.

11. Find all the spherical solutions of the three-dimensional wave equation; that is, find the solutions that depend only on r and t. (*Hint:* See (5).)

12. Solve the three-dimensional wave equation in $\{r \neq 0, t > 0\}$ with zero initial conditions and with the limiting condition

$$\lim_{r \to 0} 4\pi r^2 u_r(r, t) = g(t).$$

Assume that $g(0) = g'(0) = g''(0) = 0$.

13. Solve the wave equation in the half-space $\{(x, y, z, t) \colon z > 0\}$ with the Neumann condition $\partial u/\partial z = 0$ on $z = 0$, and with initial data $\phi(x, y, z) \equiv 0$ and general $\psi(x, y, z)$. (*Hint:* See (3) and use the method of reflection.)

14. Why doesn't the method of spherical means work for two-dimensional waves?

15. Obtain the general solution formula (19) in two dimensions from the special case (18).

16. (a) Solve the wave equation in two dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Do not carry out the integral.
    (b) Under the same conditions find a simple formula for the solution $u(\mathbf{0}, t)$ at the origin by carrying out the integral.

17. Use the result of Exercise 16 to compute the limit of $t \cdot u(\mathbf{0}, t)$ as $t \to \infty$.

18. For any solution of the two-dimensional wave equation with initial data vanishing outside some circle, prove that $u(x, y, t) = O(t^{-1})$ for fixed (x, y) as $t \to \infty$; that is, $t \cdot u(x, y, t)$ is a bounded function of t for fixed x and y. Note the contrast to three dimensions. (*Hint:* Use formula (19).)

19. (*difficult*) Show, however, that if we are interested in uniform convergence, that $u(x, y, t) = O(t^{-1/2})$ *uniformly* as $t \to \infty$.

20. "Descend" from two dimensions to one as follows. Let $u_{tt} = c^2 u_{xx}$ with initial data $\phi(x) \equiv 0$ and general $\psi(x)$. Imagine that we don't know d'Alembert's solution formula. Think of $u(x, t)$ as a solution of the two-dimensional equation that happens not to depend on y. Plug it into (19) and carry out the integration.

9.3 Rays，singularities，and sources

Characteristic

Relativistic geometry

Singularities

Wave length with a source

Exercises

1. Let S be a characteristic surface for which $S \cap \{(x, y, z)\colon t = 0\}$ is the sphere $\{x^2 + y^2 + z^2 = a^2\}$. Describe S geometrically.

2. Prove the converse of Theorem 1. That is, prove that a level surface of $t - \gamma(\mathbf{x})$ is characteristic if $\gamma(\mathbf{x})$ satisfies the nonlinear PDE

$$|\nabla\gamma(\mathbf{x})| \equiv \frac{1}{c}. \qquad (*)$$

(*Hint:* Differentiate the equation $(*)$ to get $\Sigma\gamma_{ij}(\mathbf{x})\gamma_j(\mathbf{x}) = 0$, where subscripts denote partial derivatives. Show that a curve, which satisfies the ODE $d\mathbf{x}/dt = c^2\nabla\gamma(\mathbf{x})$, also satisfies $d^2\mathbf{x}/dt^2 = 0$ and hence is a ray. Show that $t - \gamma(\mathbf{x})$ is constant along a ray. Deduce that any level surface of $t - \gamma(\mathbf{x})$ is characteristic.)

3. Prove Theorem 2 in the one-dimensional case. That is, if $\mathscr{C}$ is a spacelike curve in the xt plane, there is a unique solution of $u_{tt} = c^2u_{xx}$ with $u = \phi$ and $\partial u/\partial n = \psi$ on $\mathscr{C}$.

4. Verify that the solution given by (5) has second derivatives which have jump discontinuities on the surface $S = \{(\mathbf{x}, t): t = \gamma(\mathbf{x})\}$.

5. Verify the correctness of (13) for the example $u(x, y, z, t) = t^2$ and $f(x, y, z, t) \equiv 2$.

6. Show that the unique solution of (9) is expressible in terms of the source operator by the simple formula (11).

7. (*difficult*) Solve $u_{tt} - c^2\Delta u = f(\mathbf{x})$, where $f(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, $f(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, A is a constant, and the initial data are identically zero. Sketch the regions in space-time that illustrate your answer. (*Hint:* Use (13) and find the volume of intersection of two balls, or use (11) and Exercise 9.2.6(b).)

8. Carry out the passage from (11) to (13) more explicitly using spherical coordinates.

9. Simplify formula (13) for the solution of $u_{tt} - c^2\Delta u = f(\mathbf{x}, t)$ in the special case that f is spherically symmetric [$f = f(r, t)$].

9.4 The diffusion and Schrodinger equation

Harmonic oscillator

Exercises

1. Find a simple formula for the solution of the three-dimensional diffusion equation with $\phi(x, y, z) = xy^2z$. (*Hint:* See Exercise 2.4.9 or 2.4.10.)

2. (a) Prove that (6) is valid for products of the form $\phi(x)\psi(y)\zeta(z)$ and hence for any finite sum of such products.
   (b) Deduce (6) for any bounded continuous function $\phi(\mathbf{x})$. You may use the fact that there is a sequence of finite sums of products as in part (a) which converges uniformly to $\phi(\mathbf{x})$.

3. Find the solution of the diffusion equation in the half-space $\{(x, y, z, t): z > 0\}$ with the Neumann condition $\partial u/\partial z = 0$ on $z = 0$. (*Hint:* Use the method of reflection.)

4. Derive the first four Hermite polynomials from scratch.

5. Show that all the Hermite polynomials are given by the formula

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

up to a constant factor.

6. Show directly from the ODE (15) that the functions $H_k(x)e^{-x^2/2}$ are mutually orthogonal on the interval $(-\infty, \infty)$. That is

$$\int_{-\infty}^{\infty} H_k(x)H_l(x)e^{-x^2}\,dx = 0 \qquad \text{for } k \neq l.$$

(*Hint:* See Section 5.3.)

7. (a) Show that if $\lambda \neq 2k + 1$, any solution of Hermite's ODE is a power series but not a polynomial.
   (b) Deduce that in this case no solution of Hermite's ODE can satisfy the condition at infinity. (*Hint:* Use the recursion relation (18) to find the behavior of a_k as $k \to \infty$. Compare with the power series expansion of e^{x^2}. Deduce that $u(x, t)$ behaves like e^{x^2} as $|x| \to \infty$.)

9.5 The hydrogen atom

Exercises

1. Verify the formulas for the first three solutions of the hydrogen atom.

2. For the hydrogen atom if $\lambda > 0$, why would you expect equation (4) not to have a solution that satisfies the condition at infinity?