Peter J. Olver

Ch10 Finite Elements and Weak Solutions

10.1 Minimization and Finite Elements

10.1.1. Let $U = \{ u(x) \in C^2[0,\pi] \mid u(0) = u(\pi) = 0 \}$ and $V = \{ v(x) \in C^1[0,\pi] \}$ both be equipped with the L^2 inner product. Let $L: U \to V$ be given by $L[u] = D[u] = u'$, and $f(x) = x - 1$. (*a*) Write out the quadratic functional $Q[u]$ given by (10.1). (*b*) Write out the associated boundary value problem (10.2). (*c*) Find the function $u_\star(x) \in U$ that minimizes $Q[u]$. What is the value of $Q[u_\star]$? (*d*) Let $W \subset U$ be the subspace spanned by $\sin x$ and $\sin 2x$. Write out the corresponding finite-dimensional minimization problem (10.8). (*e*) Find the function $w_\star(x) \in W$ that minimizes $Q[w]$. Is $Q[w_\star] \geq Q[u_\star]$? If not, why not? How close is your finite element minimizer $w_\star(x)$ to the actual minimizer $u_\star(x)$?

10.1.2. Let $U = \{ u(x) \in C^2[0,1] \mid u(0) = u(1) = 0 \}$ and $V = \{ v(x) \in C^1[0,1] \}$ both have the L^2 inner product. Let $L: U \to V$ be given by $L[u] = u'(x) - u(x)$, and $f(x) = 1$ for all x. (*a*) Write out the quadratic functional $Q[u]$ given by (10.1). (*b*) Write out the associated boundary value problem (10.2). (*c*) Find the function $u_\star(x) \in U$ that minimizes $Q[u]$. What is the value of $Q[u_\star]$? (*d*) Let $W \subset U$ be the subspace containing all cubic polynomials $p(x)$ that satisfy the boundary conditions: $p(0) = p(1) = 0$. Find a basis of W and then write out the corresponding finite-dimensional minimization problem (10.8). (*e*) Find the polynomial $p_\star(x) \in W$ that minimizes $Q[p]$ for $p \in W$. Is $Q[p_\star] \geq Q[u_\star]$? If not, why not? How close is your finite element minimizer $p_\star(x)$ to the minimizer $u_\star(x)$?

10.1.3. Let $U = \{ u(x) \in C^2[1,2] \mid u(1) = u(2) = 0 \}$, $V = \{ (v_1(x), v_2(x))^T \mid v_1, v_2 \in C^1[1,2] \}$, both be endowed with the L^2 inner product. Let $L: U \to V$ be given by $L[u] = \begin{pmatrix} x\,u'(x) \\ \sqrt{2}\,u(x) \end{pmatrix}$, and let $f(x) = 2$ for all $1 \leq x \leq 2$. (*a*) Write out the quadratic functional $Q[u]$ given by (10.1). (*b*) Write out the associated boundary value problem (10.2). (*c*) Find the function $u_\star(x) \in U$ that minimizes $Q[u]$. What is the value of $Q[u_\star]$? (*d*) Let $W \subset U$ be the subspace containing all cubic polynomials $p(x)$ that satisfy the boundary conditions $p(1) = p(2) = 0$. Find a basis of W and then write out the corresponding finite-dimensional minimization problem (10.8). (*e*) Find the polynomial $p_\star(x) \in W$ that minimizes $Q[p]$ for $p \in W$. Is $Q[p_\star] \geq Q[u_\star]$? If not, why not? How close is your finite element minimizer $p_\star(x)$ to the actual minimizer $u_\star(x)$?

10.1.4. (*a*) Find the solution to the boundary value problem $-u'' = x^2 - x$, $u(-1) = u(1) = 0$. (*b*) Write down a quadratic functional $Q[u]$ that is minimized by your solution. (*c*) Let W be the subspace spanned by the two functions $(1-x^2)$, $x(1-x^2)$. Find the function $w_\star(x) \in W$ that minimizes the restriction of your quadratic functional to W. Compare $w_\star$ with your solution from part (*a*). (*d*) Answer part (*c*) for the subspace W spanned by $\sin \pi x, \sin 2\pi x$. Which of the two approximations is the better?

10.1.5. (*a*) Find the function $u_\star(x)$ that minimizes $Q[u] = \int_0^1 \left[\frac{1}{2}(x+1)u'(x)^2 - u(x) \right] dx$ over the vector space U consisting of C^2 functions satisfying $u(0) = u(1) = 0$. (*b*) Let $W_3 \subset U$ be the subspace consisting of all cubic polynomials $w(x)$ that satisfy the same boundary conditions. Find the function $w_\star(x)$ that minimizes the restriction $Q[w]$ for $w \in W_3$. Compare $w_\star(x)$ and $u_\star(x)$: how close are they in the L^2 norm? What is the maximal discrepancy $|\, w_\star(x) - u_\star(x) \,|$ for $0 \leq x \leq 1$? (*c*) Suppose you enlarge your finite-dimensional subspace $W_4 \subset U$ to contain all quartic polynomials that satisfy the boundary conditions. Is your new finite element approximation better? Discuss.

10.1.6. (a) Find the function $u_\star(x)$ that minimizes $Q[u] = \int_0^1 \left[\frac{1}{2} e^x u'(x)^2 - 3\,u(x)\right] dx$ over the space U consisting of C^2 functions satisfying the boundary conditions $u(0) = u'(1) = 0$. (b) Let $W \subset U$ be the subspace containing all cubic polynomials $w(x)$ that satisfy the boundary conditions. Find the polynomial $w_\star(x)$ that minimizes the restriction $Q[w]$ for $w \in W$. Compare $w_\star(x)$ and $u_\star(x)$: how close are they in the L^2 norm? What is the maximal discrepancy $|\,w_\star(x) - u_\star(x)\,|$ for $0 \le x \le 1$?

10.1.7. Consider the Dirichlet boundary value problem

$$-\Delta u = x\,(1-x) + y\,(1-y), \qquad u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0,$$

on the unit square $\{0 < x, y < 1\}$.

(a) Find the exact solution $u_\star(x,y)$. *Hint*: It is a polynomial.

(b) Write down a minimization principle $Q[u]$ that characterizes the solution. Be careful to specify the function space U over which the minimization takes place.

(c) Let $W \subset U$ be the subspace spanned by the four functions $\sin \pi x \sin \pi y$, $\sin 2\pi x \sin \pi y$, $\sin \pi x \sin 2\pi y$, and $\sin 2\pi x \sin 2\pi y$. Find the function $w_\star \in W$ that minimizes the restriction of $Q[w]$ to $w \in W$. How close is $w_\star$ to the solution you found in part (a)?

10.1.8. Justify the identification of (10.4) with the quadratic function (10.5).

10.2 Finite Elements for ODE

10.2.1. Use the finite element method to approximate the solution to the boundary value problem $-\dfrac{d}{dx}\left(e^{-x}\dfrac{du}{dx}\right) = 1$, $u(0) = u(2) = 0$. Carefully explain how you are setting up the calculation. Plot the resulting solutions and compare your answer with the exact solution. You should use an equally spaced mesh, but try at least three different mesh spacings and compare your results. By inspecting the errors in your various approximations, can you predict how many nodes would be required for six-digit accuracy of the numerical approximation?

10.2.2. For each of the following boundary value problems: (*i*) Solve the problem exactly. (*ii*) Approximate the solution using the finite element method based on ten equally spaced nodes. (*iii*) Compare the graphs of the exact solution and its piecewise affine finite element approximation. What is the maximal error in your approximation at the nodes? on the entire interval?

(a) $-u'' = \begin{cases} 1 & x > 1, \\ 0, & x < 1, \end{cases}$ $u(0) = u(2) = 0$; (b) $-\dfrac{d}{dx}\left((1+x)\dfrac{du}{dx}\right) = 1$, $u(0) = u(1) = 0$;

(c) $-\dfrac{d}{dx}\left(x^2\dfrac{du}{dx}\right) = -x$, $u(1) = u(3) = 0$; (d) $-\dfrac{d}{dx}\left(e^x\dfrac{du}{dx}\right) = e^x$, $u(-1) = u(1) = 0$.

10.2.3. (a) Find the exact solution to the boundary value problem $-u'' = 3x$, $u(0) = u(1) = 0$. (b) Use the finite element method based on five equally spaced nodes to approximate the solution. (c) Compare the graphs of the exact solution and its piecewise affine finite element approximation. (d) What is the maximal error (*i*) at the nodes? (*ii*) on the entire interval?

10.2.4. Use finite elements to approximate the solution to the Sturm–Liouville boundary value problem $-u'' + (x+1)u = x\,e^x$, $u(0) = 0$, $u(1) = 0$, using 5, 10, and 20 equally spaced nodes.

10.2.5. (a) Devise a finite element scheme for numerically approximating the solution to the mixed boundary value problem

$$-\frac{d}{dx}\left(\kappa(x)\,\frac{du}{dx}\right)=f(x), \qquad a<x<b, \qquad u(a)=0, \qquad u'(b)=0.$$

(b) Test your method on the particular boundary value problem

$$-\frac{d}{dx}\left((1+x)\,\frac{du}{dx}\right)=1, \qquad 0<x<1, \qquad u(0)=0, \qquad u'(1)=0,$$

using 10 equally spaced nodes. Compare your approximation with the exact solution.

10.2.6. Consider the periodic boundary value problem

$$-u''+u=x, \qquad u(0)=u(2\pi), \qquad u'(0)=u'(2\pi).$$

(a) Write down the analytic solution. (b) Write down a minimization principle. (c) Divide the interval $[0,2\pi]$ into $n=5$ equal subintervals, and let W_n denote the subspace consisting of all piecewise affine functions that satisfy the boundary conditions. What is the dimension of W_n? Write down a basis. (d) Construct the finite element approximation to the solution to the boundary value problem by minimizing the functional from part (b) on the subspace W_n. Graph the result and compare with the exact solution. What is the maximal error on the interval? (e) Repeat part (d) for $n=10,20$, and 40 subintervals, and discuss the convergence of your solutions.

10.2.7. Answer Exercise 10.2.6 when the finite element subspace W_n consists of all periodic piecewise affine functions of period 1, so $w(x+1)=w(x)$. Which approximation is better?

10.2.8. Use the method of Exercise 10.2.7 to approximate the solution to the following periodic boundary value problem for the *Mathieu equation*:

$$-u''+(1+\cos x)\,u=1, \qquad u(0)=u(2\pi), \qquad u'(0)=u'(2\pi).$$

10.2.9. Consider the boundary value problem solved in Example 10.3. Let W_n be the subspace consisting of all polynomials $u(x)$ of degree $\leq n$ satisfying the boundary conditions $u(0)=u(1)=0$. In this project, we will try to approximate the exact solution to the boundary value problem by minimizing the functional (10.24) on the polynomial subspace W_n. For $n=5,10$, and 20: (a) First, determine a basis for W_n. (b) Set up the minimization problem as a system of linear equations for the coefficients of the polynomial minimizer relative to your basis. (c) Solve the polynomial minimization problem and compare your "polynomial finite element" solution with the exact solution and the piecewise affine finite element solution graphed in Figure 10.3.

10.2.10. Consider the boundary value problem $-u''+\lambda u=x$, for $0<x<\pi$, with $u(0)=0$, $u(1)=0$. (a) For what values of λ does the system have a unique solution? (b) For which values of λ can you find a minimization principle that characterizes the solution? Is the minimizer unique for all such values of λ? (c) Using n equally spaced nodes, write down the finite element equations for approximating the solution to the boundary value problem. *Note*: Although the finite element construction is supposed to work only when there is a minimization principle, we will consider the resulting linear algebraic system for any value of λ. (d) Select a value of λ for which the solution can be characterized by a minimization principle and verify that the finite element approximation with $n=10$ approximates the exact solution. (e) Experiment with other values of λ. Does your finite element solution give a good approximation to the exact solution when it exists? What happens at values of λ for which the solution does not exist or is not unique?

10.3 Finite Elements in Two Dimensions

10.3.1. Sketch a triangulation of the following domains so that all triangles have side length at most .5: (*a*) a unit square; (*b*) an isosceles triangle with vertices $(-.5,0),(.5,0)$ and $(0,1)$; (*c*) the square $\{\,|\,x\,|,|\,y\,|\leq 2\}$ with the hole $\{\,|\,x\,|,|\,y\,|<1\}$ removed; (*d*) the unit disk; (*e*) the annulus $1\leq\|\,\mathbf{x}\,\|\leq 2$.

10.3.2. Describe the vertex polygons for a triangulation that uses regular equilateral triangles.

10.3.3. Are there any restrictions on the number of sides a vertex polygon can have?

10.3.4. Find the three finite element functions $\omega_1(x,y)$, $\omega_2(x,y)$, $\omega_3(x,y)$, associated with
(*a*) the triangle having vertices $(1,0),(0,1)$, and $(1,1)$;
(*b*) the triangle having vertices $(0,1),(1,-1)$, and $(-1,-1)$;
(*c*) an equilateral triangle centered at the origin having one vertex at $(1,0)$.

10.3.5. (*a*) Prove that the area of a planar triangle T with vertices $(a,b),(c,d),(e,f)$ is equal to $\frac{1}{2}\,|\,\Delta\,|$, where $\Delta=\det\begin{pmatrix}1 & a & b\\ 1 & c & d\\ 1 & e & f\end{pmatrix}$. (*b*) Prove that $\Delta>0$ if and only if the vertices of the triangle are listed in counterclockwise order.

10.3.6. Give a detailed justification of the continuity of the pyramid function (10.36).

10.3.7. An alternative to triangular elements is to employ piecewise *bi-affine functions*, meaning $\omega(x,y)=\alpha+\beta\,x+\gamma\,y+\delta\,x\,y$, on rectangles. (*a*) Suppose R is a rectangle with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$, whose sides are parallel to the coordinate axes. Prove that, for each $l=1,\ldots,4$, there is a unique bi-affine function $\omega_l(x,y)$ defined on R that has the value $\omega_l(x_l,y_l)=1$ at one vertex while $\omega_l(x_i,y_i)=0,\ i\neq l$, at the other three vertices.

(*b*) Write out the four bi-affine functions $\omega_1(x,y),\ldots,\omega_4(x,y)$, when
(*i*) $R=\{0\leq x,y\leq 1\}$, (*ii*) $R=\{-1\leq x,y\leq 1\}$. (*c*) Does the result in part (*a*) hold for rectangles whose sides are not aligned with the axes? For general quadrilaterals?

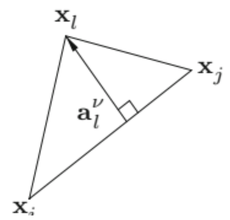
The Finite Element Equations

10.3.8. Write down the elemental stiffnesses for: (*a*) the triangle with vertices $(0,1),(-1,2),(0,-1)$; (*b*) the triangle with vertices $(1,1),(-1,1),(0,-2)$; (*c*) a $30-60-90$ degree right triangle; (*d*) a right triangle with side lengths $3,4,5$; (*e*) an isosceles triangle of height 3 and base 2; (*f*) a "golden" isosceles triangle with angles $36^\circ,72^\circ,72^\circ$.

10.3.9. A *rectangular mesh* has nodes $\mathbf{x}_{i,j}=(i\,\Delta x+a,j\,\Delta y+b)$, where $\Delta x,\Delta y>0$ are, respectively, the horizontal and vertical step sizes. Find the elemental stiffnesses for the triangles associated with such a rectangular mesh.

10.3.10. *True or false*: Let T be a triangle, and $\widetilde{T}$ a triangle obtained by rotating T by $60°$. Then T and $\widetilde{T}$ have the same elemental stiffnesses.

10.3.11. Prove that the gradient (10.42) of the affine element is equal to $\nabla \omega_l^\nu = \| \mathbf{a}_l^\nu \|^{-2} \mathbf{a}_l^\nu$, where $\mathbf{a}_l^\nu$ is the *altitude vector* that goes to the vertex $\mathbf{x}_l$ from its opposite side, as indicated in the figure.



10.3.12. Explain why the pyramid functions are linearly independent.

10.3.13. Prove formulas (10.46).

Assembling the Elements

The Coefficient Vector and the Boundary Conditions

Inhomogeneous Boundary Conditions

10.3.14. Consider the Dirichlet boundary value problem $\Delta u = 0$, $u(x,0) = \sin x$, $u(x,\pi) = 0$, $u(0,y) = 0$, $u(\pi,y) = 0$, on the square $S = \{0 < x, y < \pi\}$. (*a*) Find the exact solution. (*b*) Set up and solve the finite element equations based on a square mesh with $n = 2$ squares on each side of S. Write out the reduced finite element matrix, the boundary coefficient matrix, and the value of your approximation at the middle of the unit square. How close is this value to the exact solution there? (*c*) Repeat part (*b*) for $n = 4$ squares per side. Is the value of your approximation at the center of the unit square closer to the true solution? (*d*) Use a computer to find a finite element approximation to $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ using $n = 8$ squares per side. Is your approximation converging to the exact solution as the mesh becomes finer and finer?

10.3.15. Approximate the solution to the Dirichlet problem $\Delta u = 0$, $u(x,0) = x$, $u(x,1) = 1 - x$, $u(0,y) = y$, $u(1,y) = 1 - y$, by use of finite elements with mesh sizes $\Delta x = \Delta y = .25$ and $.1$. Compare your approximations with the solution you obtained in Exercise 4.3.12(*d*). What is the maximal error at the nodes in each case?

10.3.16. A metal plate has the shape of an equilateral triangle with unit sides. One side is heated to $100°$, while the other two are kept at $0°$. In order to approximate the equilibrium temperature distribution, the plate is divided into smaller equilateral triangles, with n triangles on each side, and the corresponding finite element approximation is then computed. (*a*) How many triangles are in the triangulation? How many interior nodes? How many edge nodes? (*b*) For $n = 2$, set up and solve the finite element linear system to find an approximation to the temperature at the center of the triangle. (*c*) Answer part (*b*) when $n = 3$. (*d*) Use a computer to find the finite element approximation to the temperature at the center when $n = 5, 10$, and 15. Are your values converging to the actual temperature? (*e*) Plot the finite element approximations you constructed in the previous parts.

10.3.17. Find the equilibrium temperature distribution in a unit equilateral triangle when one side is heated to $100°$, while the other two are insulated.

10.3.18. A metal plate has the shape of a 3 cm square with a 1 cm square hole cut out of the middle. The plate is heated by fixing the inner edge at temperature $100°$ while keeping the outer edge at $0°$. (*a*) Find the (approximate) equilibrium temperature using finite elements with a mesh width of $\Delta x = \Delta y = .5$ cm. Plot your approximate solution using a three-dimensional graphics program. (b) Let C denote the square contour lying midway between the inner and outer square boundaries of the plate. Using your finite element approximation, at what point(s) on C is the temperature a (i) minimum? (ii) maximum? (iii) equal to $50°$, the average of the two boundary temperatures? (c) Repeat part (a) using a smaller mesh width of $h = .2$. How much does this affect your answers in part (b)?

10.3.19. Answer Exercise 10.3.18 when the plate is additionally subjected to a constant heat source $f(x,y) = 600\,x + 800\,y - 2400$.

10.3.20. (a) Construct a finite element approximation to the solution, using a maximal mesh size of .1, to the following boundary value problem on the unit disk:

$$\Delta u = 0, \qquad x^2 + y^2 < 1, \qquad u = \begin{cases} 1, & x^2 + y^2 = 1, \ \ y > 0, \\ 0, & x^2 + y^2 = 1, \ \ y < 0. \end{cases}$$

(b) Compare your solution with the exact solution given in Example 4.7.

10.3.21. (a) Use finite elements to approximate the solution to the boundary value problem

$$-\,\Delta u + u = 0, \qquad 0 < x, y < 1, \qquad u(x,0) = u(x,1) = u(0,y) = 0, \qquad u(1,y) = 1.$$

(b) Compare your result with the first 5 and 10 summands in the series solution obtained via separation of variables.

10.3.21. (a) Use finite elements to approximate the solution to the boundary value problem

$$-\,\Delta u + u = 0, \qquad 0 < x, y < 1, \qquad u(x,0) = u(x,1) = u(0,y) = 0, \qquad u(1,y) = 1.$$

(b) Compare your result with the first 5 and 10 summands in the series solution obtained via separation of variables.

10.3.22. (a) Justify the construction of the finite element matrix for a square mesh described in the text. (b) How would you modify the matrix for a rectangular mesh, as in Exercise 10.3.9?

10.3.23. Justify the inhomogeneous finite element construction in the text.

10.3.24. (a) Explain how to adapt the finite element method to a mixed boundary value problem with inhomogeneous Neumann conditions. (b) Apply your method to the problem

$$\Delta u = 0, \qquad \frac{\partial u}{\partial y}(x,0) = x, \qquad u(x,1) = 0, \qquad u(0,y) = 0, \qquad u(1,y) = 0.$$

(c) Solve the boundary value problem via separation of variables. Compare the values of your solutions at the center of the square.

10.4 Weak Solutions

10.4.1. Write out semi-weak and fully weak formulations for the following boundary value problems: (a) $-u'' + 2u = x - x^2$, $u(0) = u(1) = 0$;
(b) $e^x\, u'' + u = \cos x$, $u'(0) = u'(2) = 0$; (c) $x\,u'' + u' + x\,u = 0$, $u(1) = u(2) = 0$.

10.4.2. (a) Write down a weak formulation for the boundary value problem $-u'' + 3u = x$, $u(0) = u(1) = 0$. (b) Based on your weak formulation, construct a finite element approximation to the solution, using $n = 10$ nodes.

10.4.3. (a) Write down a weak formulation of the transport equation $u_t + 3u_x = 0$ on the real line. (b) Solve the initial value problem $u(0,x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$
(c) Explain why the result of part (b) is not a classical solution to the wave equation. Is it a weak solution according to your formulation in part (a)?

10.4.4. (a) Write down a semi-weak formulation of the wave equation $u_{tt} = 4u_{xx}$ on the real line. (b) Solve the initial value problem $u(0,x) = \rho(x)$, $u_t(0,x) = 0$, where the initial displacement is a ramp function (6.25). (c) Explain why the result of part (b) is not a classical solution to the wave equation. Does it satisfy the semi-weak formulation of part (a)? Explain your answer.

10.4.5. (a) Starting with the nonlinear transport equation written in the alternative conservative form (2.56), find a corresponding weak formulation.
(b) Prove that your weak formulation produces the alternative entropy condition (2.58) for the motion of a shock discontinuity.

10.4.6. Prove that the du Bois–Reymond Lemma 10.12 remains valid even when $v(x) \in C^\infty$ is required to be infinitely differentiable.

10.4.7. *The Two-dimensional du Bois–Reymond Lemma*: Let $\Omega \subset \mathbb{R}^2$ be a domain, and $f(t,x)$ a continuous function defined thereon. Prove that $\iint_\Omega f(t,x)\, v(t,x)\, dt\, dx = 0$ for every C^1 function $v(t,x)$ with compact support in Ω if and only if $f(t,x) \equiv 0$.

10.4.8. (a) Investigate the ability of finite elements to approximate a solution to the non-positive-definite boundary value problem $\Delta u + \lambda u = 0$, $0 < x < \pi$, $0 < y < \pi$, $u(x,0) = 1$, $u(x,\pi) = u(0,y) = u(\pi,y) = 0$, when (i) $\lambda = 1$, (ii) $\lambda = 2$. Use separation of variables to find a series solution and use it to determine the accuracy of your finite element solution in part (a).