# ***Students' Selected Solutions Manual***

for

# *Introduction to Partial Differential Equations*

by

Peter J. Olver

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# Student Solutions to

## Chapter 1: What Are Partial Differential Equations?

1.1. (a) Ordinary differential equation, equilibrium, order = 1;

(c) partial differential equation, dynamic, order = 2;

(e) partial differential equation, equilibrium, order = 2;

1.2. (a) (i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, (ii) $u_{xx} + u_{yy} = 0$.

1.4. (a) independent variables: $x, y$; dependent variables: $u, v$; order = 1.

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1.5. (a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$; defined and $\mathrm{C}^\infty$ on all of $\mathbb{R}^2$.

1.7. $u = \log\big[c\,(x-a)^2 + c\,(y-b)^2\big]$, for $a, b, c$ arbitrary constants.

1.10. (a) $\dfrac{\partial^2 u}{\partial t^2} - 4\,\dfrac{\partial^2 u}{\partial x^2} = 8 - 8 = 0$.

1.11. (a) $c_0 + c_1 t + c_2 x + c_3 (t^2 + x^2) + c_4 t x$, where $c_0, \ldots, c_4$ are arbitrary constants.

1.15. Example: (b) $u_x^2 + u_y^2 + u^2 = 0$ — the only real solution is $u \equiv 0$.

---

1.17. (a) homogeneous linear; (d) nonlinear.

1.19. (a) (i) $\dfrac{\partial^2 u}{\partial t^2} = -4\cos(x - 2t) = 4\,\dfrac{\partial^2 u}{\partial x^2}$.

1.21. (a) $\partial_x[c f + d g] = \dfrac{\partial}{\partial x}\,[c f(x) + d g(x)] = c\,\dfrac{\partial f}{\partial x} + d\,\dfrac{\partial g}{\partial x} = c\,\partial_x[f] + d\,\partial_x[g]$. The same proof works for $\partial_y$. (b) Linearity requires $d = 0$, while $a, b, c$ can be arbitrary functions of $x, y$.

1.24. (a)
$$\begin{aligned}(L - M)[u + v] &= L[u + v] - M[u + v] = L[u] + M[u] - L[v] - M[v] \\ &= (L - M)[u] + (L - M)[v],\end{aligned}$$
$$(L - M)[c u] = L[c u] - M[c u] = c L[u] - c M[u] = c(L - M)[u].$$

1.27. (b) $u(x) = \frac{1}{6}\, e^x \sin x + c_1\, e^{2x/5} \cos \frac{4}{5}\, x + c_2\, e^{2x/5} \sin \frac{4}{5}\, x$.

1.28. (b) $u(x) = -\frac{1}{9}\, x - \frac{1}{10} \sin x + c_1\, e^{3x} + c_2\, e^{-3x}$.

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# Student Solutions to

## Chapter 2: Linear and Nonlinear Waves

2.1.3. (a) $u(t,x) = f(t)$; (e) $u(t,x) = e^{-t\,x} f(t)$.

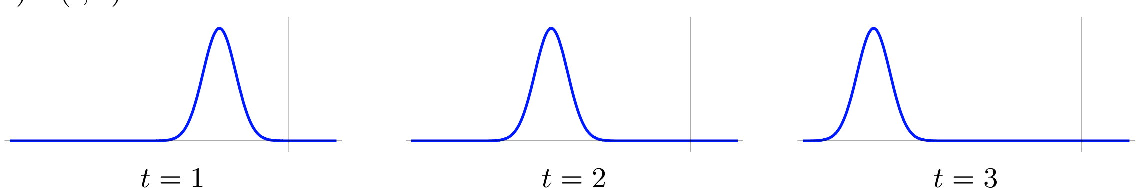
2.1.5. $u(t,x,y) = f(x,y)$ where $f$ is an arbitrary $\mathrm{C}^1$ function of two variables. This is valid provided each slice $D_{a,b} = D \cap \{\,(t,a,b) \mid t \in \mathbb{R}\,\}$, for fixed $(a,b) \in \mathbb{R}^2$, is either empty or a connected interval.

♢ 2.1.9. It suffices to show that, given two points $(t_1,x), (t_2,x) \in D$, then $u(t_1,x) = u(t_2,x)$. By the assumption, $(t,x) \in D$ for $t_1 \le t \le t_2$, and so $u(t,x)$ is defined and continuously differentiable at such points. Thus, by the Fundamental Theorem of Calculus,

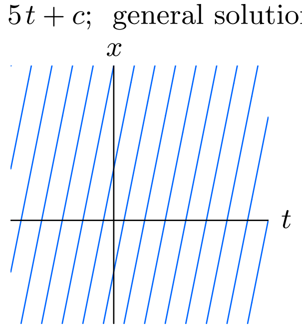
$$u(t_2,x) - u(t_1,x) = \int_{t_1}^{t_2} \frac{\partial u}{\partial t}(s,x)\,ds = 0. \qquad \textit{Q.E.D.}$$

---

2.2.2. (a) $u(t,x) = e^{-(x+3\,t)^2}$




2.2.3. (b) Characteristic lines: $x = 5\,t + c$; general solution: $u(t,x) = f(x - 5\,t)$;




♢ 2.2.6. By the chain rule

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t - t_0, x), \qquad \frac{\partial v}{\partial x}(t,x) = \frac{\partial u}{\partial x}(t - t_0, x),$$

and hence

$$\frac{\partial v}{\partial t}(t,x) + c\,\frac{\partial v}{\partial x}(t,x) = \frac{\partial u}{\partial t}(t - t_0, x) + c\,\frac{\partial u}{\partial x}(t - t_0, x) = 0.$$

Moreover, $v(t_0,x) = u(0,x) = f(x)$. *Q.E.D.*

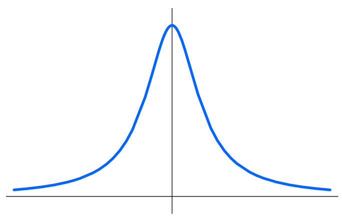
2.2.14. (a) $u(t,x) = \begin{cases} f(x - c\,t), & x \ge c\,t, \\ g(t - x/c), & x \le c\,t, \end{cases}$ defines a classical $\mathrm{C}^1$ solution provided the compatibility conditions $g(0) = f(0)$, $g'(0) = -c\,f(0)$, hold.

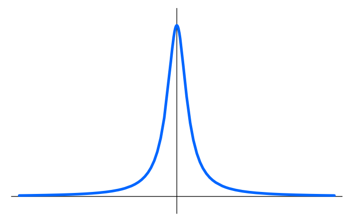
(b) The initial condition affects the solution for $x \geq c\,t$, whereas the boundary condition affects the solution for $x \leq c\,t$. Apart from the compatibility condition along the characteristic line $x = c\,t$, they do not affect each other.

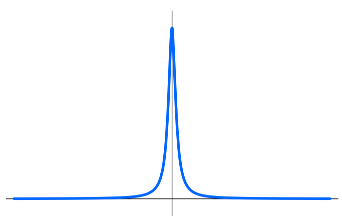
---

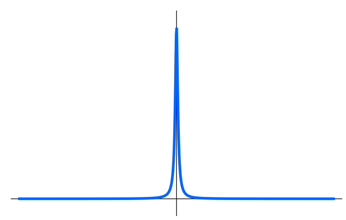
2.2.17. (a) $u(t,x) = \dfrac{1}{(x\,e^t)^2 + 1} = \dfrac{e^{-2t}}{x^2 + e^{-2t}}$.

(b)

$t = 0$: 

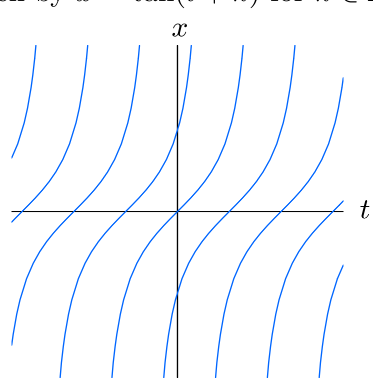
$t = 1$: 

$t = 2$: 

$t = 3$: 

(c) The limit is discontinuous: $\displaystyle \lim_{t \to \infty} u(t,x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

2.2.20. (a) The characteristic curves are given by $x = \tan(t + k)$ for $k \in \mathbb{R}$.




(b) The general solution is $u(t,x) = g(\tan^{-1} x - t)$, where $g(\xi)$ is an arbitrary $\mathrm{C}^1$ function of the characteristic variable.

(c) The solution is $u(t,x) = f\big(\tan(\tan^{-1} x - t)\big)$. Observe that the solution is not defined for $x < \tan\big(t - \frac{1}{2}\pi\big)$ for $0 < t < \pi$, nor at any value of $x$ after $t \geq \pi$. As $t$ increases up to $\pi$, the wave moves rapidly off to $+\infty$ at an ever accelerating rate, and the solution effectively disappears.

♡ 2.2.26. (a) Suppose $x = x(t)$ solves $\dfrac{dx}{dt} = c(t,x)$. Then, by the chain rule,

$$\frac{d}{dt}\,u\big(t,x(t)\big) = \frac{\partial u}{\partial t}\,\big(t,x(t)\big) + \frac{\partial u}{\partial x}\,\big(t,x(t)\big)\,\frac{dx}{dt} = \frac{\partial u}{\partial t}\,\big(t,x(t)\big) + c\big(t,x(t)\big)\,\frac{\partial u}{\partial x}\,\big(t,x(t)\big) = 0,$$

since we are assuming that $u(t,x)$ is a solution to the transport equation for all $(t,x)$. We conclude that $u\big(t,x(t)\big)$ is constant.

2.2.27. (a) The characteristic curves are the cubics $x = \frac{1}{3}t^3 + k$, where $k$ is an arbitrary constant.

(b) The solution $u(t,x) = e^{-\,(x-t^3/3)^2}$ is a Gaussian hump of a fixed shape that comes in from the left for $t < 0$, slowing down in speed as $t \to 0^-$, stops momentarily at the origin at $t = 0$, but then continues to move to the right, accelerating as $t \to \infty$.

---

2.3.1. (a) $u(t,x) = \begin{cases} 2, & x < \frac{3}{2}t - 1, \\ 1, & x > \frac{3}{2}t - 1, \end{cases}$

is a shock wave moving to the right with speed $\frac{3}{2}$ and jump magnitude 1.

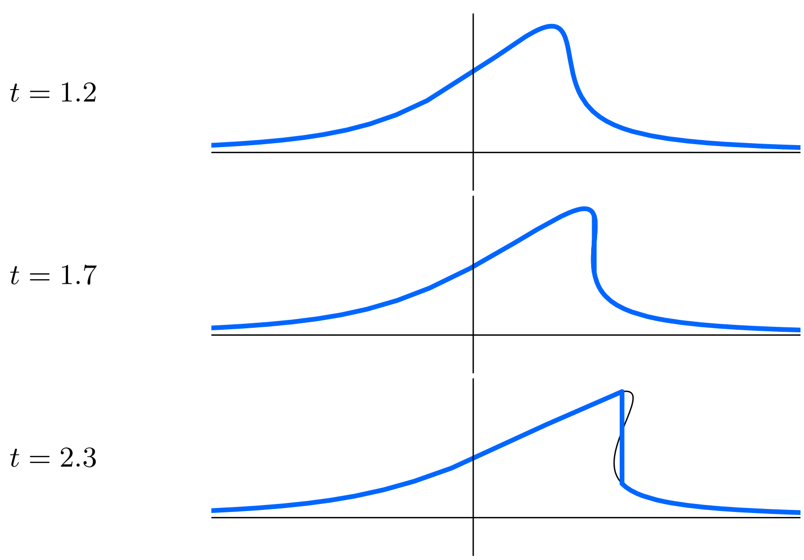
2.3.3. Yes, a shock wave is produced. According to (2.41), when $f(x) = (x^2+1)^{-1}$, the shock starts at time

$$t_\star = \min\left\{ \frac{(x^2+1)^2}{2x} = \tfrac{1}{2}x^3 + x + \tfrac{1}{2}x^{-1} \;\middle|\; x > 0 \right\} = \frac{8}{3\sqrt{3}} \approx 1.5396.$$

The minimum value occurs at $x_\star = 1/\sqrt{3}$, which is found by setting the derivative

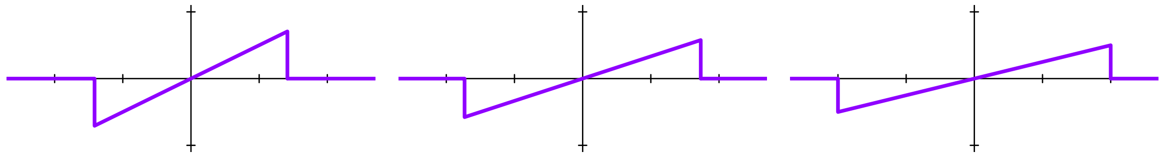
$$\frac{d}{dx}\left(\tfrac{1}{2}x^3 + x + \tfrac{1}{2}x^{-1}\right) = \tfrac{3}{2}x^2 + 1 - \tfrac{1}{2}x^{-2} = 0.$$

The solution is graphed at the indicated times:




2.3.6. (a) If and only if $\alpha = \gamma$.

2.3.9. (b) $u(t,x) = \begin{cases} x/(t+1), & -\sqrt{t+1} < x < \sqrt{t+1}\,, \\ 0, & \text{otherwise}, \end{cases}$



$t = .5$ $t = 1$ $t = 2$

The mass is conserved because the area under the graph of the solution at each time is constant, namely 0.

♢ 2.3.14. (a) According to the Implicit Function Theorem, the equation

$$F(t,x,u) = u - f(x - t\,u) = 0$$

can be locally uniquely solved for $u(t,x)$ provided

$$0 \neq \frac{\partial F}{\partial u} = 1 + t\,f'(x - t\,u), \qquad \text{and so} \qquad f'(x - t\,u) \neq -\frac{1}{t}\,.$$

2.3.15. It is a solution if and only if either

$(i)$ $k = 1/2$ and $\alpha = \gamma$, or $(ii)$ $k = 0$, or $(iii)$ $\alpha = \gamma = 0$, or $(iv)$ $\alpha = \beta = 0$.

2.3.17. (a) The mass conservation law is

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{3}\,u^3\right) = 0,$$

and so, following the previous argument, the shock speed is given by

$$\frac{d\sigma}{dt} = \frac{\frac{1}{3}\,[\,u^-(t)^3 - u^+(t)^3\,]}{u^-(t) - u^+(t)} = \frac{u^-(t)^2 + u^-(t)\,u^+(t) + u^+(t)^2}{3}\,.$$

$(b)$ $(i)$ If $|\,a\,| > |\,b\,|$, then we have a shock wave solution:

$$u(t,x) = \begin{cases} a, & x < c\,t, \\ b, & x > c\,t, \end{cases} \qquad \text{where} \qquad c = \frac{a^2 + a\,b + b^2}{3}\,.$$

Note that, in this case, $c > 0$ and so shocks always move to the right.

$(ii)$ On the other hand, if $|\,a\,| < |\,b\,|$, then we have a rarefaction wave:

$$u(t,x) = \begin{cases} a, & x \leq a^2 t, \\ \sqrt{x/t}\,, & a^2 t \leq x \leq b^2 t, \\ b, & x \geq b^2 t. \end{cases}$$

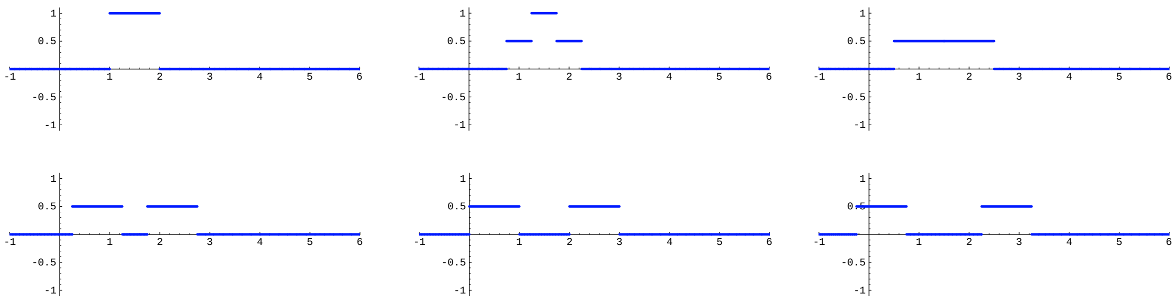
---

2.4.2. (a) The initial displacement splits into two half sized replicas, moving off to the right and to the left with unit speed.

For $t < \frac{1}{2}$, we have $u(t,x) = \begin{cases} 1, & 1 + t < x < 2 - t, \\ \frac{1}{2}, & 1 - t < x < 1 + t \quad \text{or} \quad 2 - t < x < 2 + t, \\ 0, & \text{otherwise}, \end{cases}$

For $t \geq \frac{1}{2}$, we have $u(t,x) = \begin{cases} \frac{1}{2}, & 1 - t < x < 2 - t \quad \text{or} \quad 1 + t < x < 2 + t, \\ 0, & \text{otherwise}, \end{cases}$

(*b*) Plotted at times $t = 0, .25, .5, .75, 1., 1.25$:



2.4.4. (*b*) $\displaystyle \frac{1}{2}\int_{x-t}^{x+t} 2\cos(2z)\,dz = \frac{\sin 2(x+t) - \sin 2(x-t)}{2}$.

2.4.8. (a) $\{\,(t,x) \mid 2-2t \le x \le 2+2t,\ t \ge 0\,\}$.

♡ 2.4.11. (a) $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t)$; (*b*) True.

♢ 2.4.13. First of all, the decay assumption implies that $E(t) < \infty$ for all $t$. To show $E(t)$ is constant, we prove that its derivative is 0. Using the smoothness of the solution to justify bringing the derivative under the integral sign, we compute

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} (\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2u_x^2)\,dx = \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt})\,dx \\ &= c^2\int_0^{\ell} (u_t u_{xx} + u_x u_{xt})\,dx = c^2\int_{-\infty}^{\infty} \frac{d}{dx}(u_t u_x)\,dx = 0,\end{aligned}$$

since $u_t, u_x \to 0$ as $x \to \infty$. *Q.E.D.*

2.4.17. (a) Because $\big(\partial_t + c(x)\,\partial_x\big)\big(\partial_t - c(x)\,\partial_x\big) = \partial_t^2 - c(x)^2\,\partial_x^2 - c(x)\,c'(x)\partial_x \ne \partial_t^2 - c(x)^2\,\partial_x^2$.

♢ 2.4.20. (a) Setting $x = r\cos\theta$, $y = r\sin\theta$, we have $dx\,dy = r\,dr\,d\theta$, and hence

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dx\,dy &= \int_{-\pi}^{\pi}\int_0^{\infty} r\,e^{-ar^2}\,dr\,d\theta \\ &= 2\pi\int_0^{\infty} r\,e^{-ar^2}\,dr = -\frac{\pi}{a}\,e^{-ar^2}\,\Big|_{r=0}^{\infty} = \frac{\pi}{a}.\end{aligned}$$

(*b*) By part (a),

$$\begin{aligned}\frac{\pi}{a} = \iint_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dx\,dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2}\,e^{-ay^2}\,dy\,dx \\ &= \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)^2.\end{aligned}$$

Taking square roots of both sides establishes the identity. *Q.E.D.*

# Student Solutions to

## Chapter 3: Fourier Series

3.1.1. (b) (i) $$\left(\frac{\partial}{\partial x}+1\right)\left[\,u(x)+v(x)\,\right]=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+u(x)+v(x)$$
$$=\left(\frac{\partial}{\partial x}+1\right)u(x)+\left(\frac{\partial}{\partial x}+1\right)v(x)\,,$$
$$\left(\frac{\partial}{\partial x}+1\right)\left[\,c\,u(x)\,\right]=c\,\frac{\partial u}{\partial x}+c\,u(x)=c\left(\frac{\partial}{\partial x}+1\right)u(x)\,;$$

(ii) $$\left(\frac{\partial}{\partial x}+1\right)\left[\,c(t)\,u(x)\,\right]=c(t)\,\frac{\partial u}{\partial x}+c(t)\,u(x)=c(t)\left(\frac{\partial}{\partial x}+1\right)u(x)\,;$$

(iii) $$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}+u.$$

3.1.2. (a) $\exp(-\,n^2\,t)\sin n\,x$ for $n=1,2,\ldots.$

3.1.4. (a) $u(t,x)=e^{\lambda\,(t+x)}$.

---

3.2.1. (a) $\displaystyle \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2\,j+1)\,x}{2\,j+1}\,;$ (b) $\displaystyle \frac{\pi}{2}-\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2\,j+1)\,x}{(2\,j+1)^2}\,.$

3.2.2. (b) $\displaystyle \frac{1}{2}-\frac{2}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j\cos(2\,j+1)\,x}{2\,j+1}\,.$

3.2.5. (a) True.

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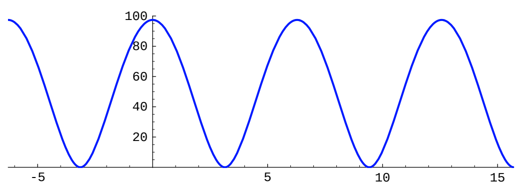
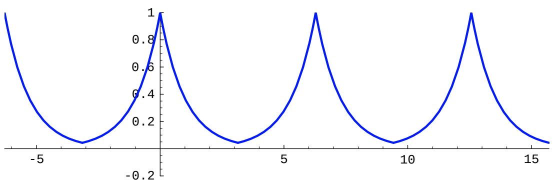
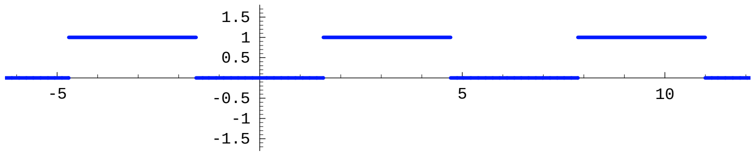
3.2.6. (b) 

differentiable,

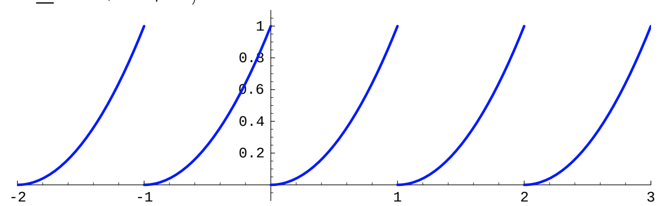
(d) 

continuous.

3.2.7. (*b*)




3.2.12. (a) $f(x) = (x-n)^2$, when $n \le x < n+1$;

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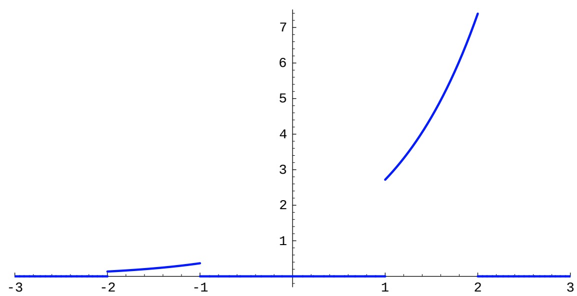
3.2.14. (a) Discontinuities: $x = -1$, magnitude 1; $x = 0$, magnitude 2; $x = 1$, magnitude $= 3$;

(*d*) no discontinuities.

3.2.15. (a) Discontinuities: $x = -2$, magnitude $e^{-2}$;
$x = -1$, magnitude $-e^{-1}$;
$x = 1$, magnitude $e$; $x = 2$, magnitude $-e^2$;




3.2.16.
3.2.14 (a) Yes: no corners; (*d*) yes: corners at $x = 0, 2$.

3.2.15 (a) Yes: no corners.

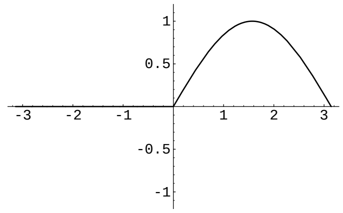
3.2.19. (a) Piecewise continuous, but not piecewise $\mathrm{C}^1$ or piecewise $\mathrm{C}^2$.

3.2.21. (a) If $f$ and $g$ are continuous at $x$, so is $f+g$. More generally, since the limit of a sum is the sum of the limits, $(f+g)(x^-) = f(x^-) + g(x^-)$, $(f+g)(x^+) = f(x^+) + g(x^+)$, and so $f+g$ is piecewise continuous at every $x$.

(*b*) Every jump discontinuity of $f$ or of $g$ is a jump discontinuity of $f+g$, except when $f$ and $g$ have opposite jump magnitudes at the same point, so $f(x^+) - f(x^-) = g(x^-) - g(x^+)$, in which case $x$ is a removable discontinuity of $f+g$. The jump magnitude of $f+g$ at $x$ is the sum of the jump magnitudes of $f$ and $g$, namely, $f(x^+) - f(x^-) + g(x^+) - g(x^-)$.

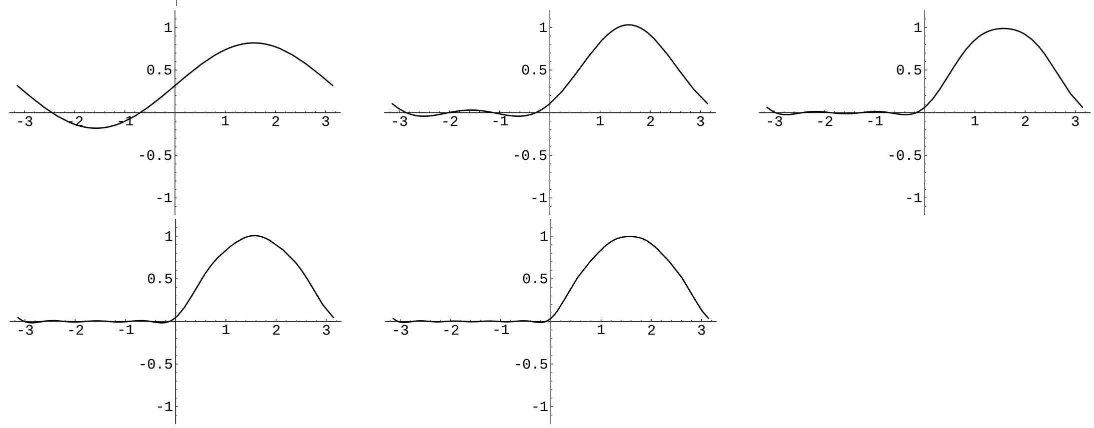
(*c*) The sum $2\sigma(x) + \sigma(x+1) - 3\sigma(x-1) + \operatorname{sign}(x^2 - 2x) = \sigma(x+1) - \sigma(x-1)$ has jump discontinuities at $x = -1$ of magnitude 1 and at $x = 1$ of magnitude $-1$. The jumps at $x = 0$ have canceled out, leaving a removable discontinuity.

3.2.25. (a)




(b) $f(x) \sim \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\displaystyle\sum_{j=1}^{\infty} \dfrac{\cos 2jx}{4j^2-1}\,.$

(c)




The maximal errors on $[-\pi,\pi]$ are, respectively .3183, .1061, .06366, .04547, .03537, .02894.

(d) The Fourier series converges (uniformly) to $\sin x$ when $2k\pi \le x \le (2k+1)\pi$ and to 0 when $(2k-1)\pi \le x \le 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$ .

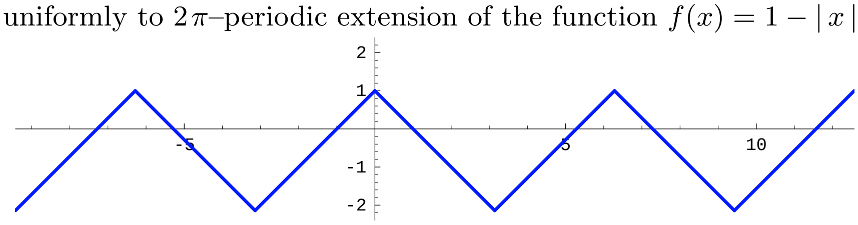
3.2.31. (a) Even, (c) odd.

♢ 3.2.33. (a) If both $f, g$ are even, then $f(-x)\,g(-x) = f(x)\,g(x)$;
if both $f, g$ are odd, then $f(-x)\,g(-x) = (-f(x))\,(-g(x)) = f(x)\,g(x)$;
if $f$ is even and $g$ is odd, then $f(-x)\,g(-x) = f(x))\,(-g(x)) = -f(x)\,g(x)$. *Q.E.D.*

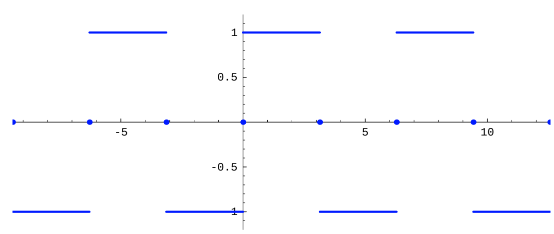
3.2.37. (a) True.

3.2.39. Even extension: $1 - \dfrac{\pi}{2} + \dfrac{4}{\pi}\displaystyle\sum_{j=0}^{\infty} \dfrac{\cos(2j+1)x}{(2j+1)^2}$;

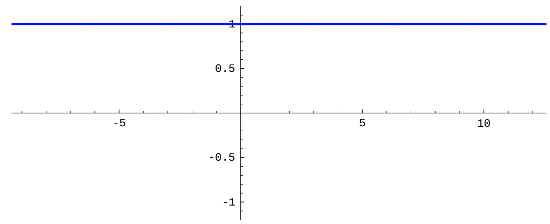
converges uniformly to $2\pi$–periodic extension of the function $f(x) = 1 - |\,x\,|$.




3.2.41. (a) Sine series: $\dfrac{4}{\pi}\displaystyle\sum_{j=0}^{\infty} \dfrac{\sin(2j+1)x}{2j+1}$;



cosine series: $1$;




3.2.42. $$\cosh m x \;\sim\; \frac{\sinh m\pi}{m\pi} \;+\; \frac{2m\sinh m\pi}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \cos k x}{k^2+m^2}.$$

---

3.2.51. (a) $\frac{1}{2}\,\mathrm{i}\, e^{-\mathrm{i}\,x} - \frac{1}{2}\,\mathrm{i}\, e^{\mathrm{i}\,x}$, (c) $\displaystyle \mathrm{i} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{(-1)^k e^{\mathrm{i}\,k\,x}}{k}$.

3.2.54. We substitute $x=\pi$ into the Fourier series (3.68) for $e^x$:

$$\tfrac{1}{2}(e^{\pi}+e^{-\pi}) = \frac{\sinh\pi}{2\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k(1+\mathrm{i}\,k)}{1+k^2}\, e^{\mathrm{i}\,k\,\pi} = \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1+\sum_{k=1}^{\infty}\frac{2}{1+k^2}\right),$$

which gives the result.

♢ 3.2.58. Replace $x$ in the Fourier series for $f(x)$ by $x-a$. Thus, the complex Fourier coefficients of $f(x-a)$ are $\widehat{c}_k = e^{-\mathrm{i}\,a}\, c_k$, where $c_k$ are the complex Fourier coefficients of $f(x)$.

---

3.3.1. (a) $$\rho(x) \;\sim\; \frac{\pi}{4} \;-\; \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)\,x}{(2j+1)^2} + \sum_{k=1}^{\infty}(-1)^{k-1}\,\frac{\sin k\,x}{k}.$$

3.3.4. (a) Integrating (3.74), we have

$$\frac{1}{6}\,x^3 - \frac{\pi^2}{6}\,x \;\sim\; -\,2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\,\sin k\,x,$$

and hence, in view of (3.73),

$$x^3 \;\sim\; 12\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{k^3}-\frac{\pi^2}{6\,k}\right)\sin k\,x.$$

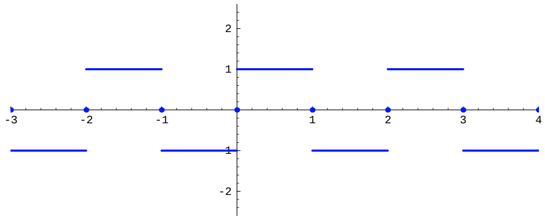
♢ 3.3.9. If $f$ is piecewise continuous and has mean zero, so $c_0=0$, then the complex Fourier series for its integral is

$$g(x) = \int_0^x f(y)\,dy \;\sim\; m \;-\sum_{0\neq k=-\infty}^{\infty} \mathrm{i}\,\frac{c_k}{k}\,e^{\mathrm{i}\,k\,x}, \qquad \text{where} \qquad m = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx.$$

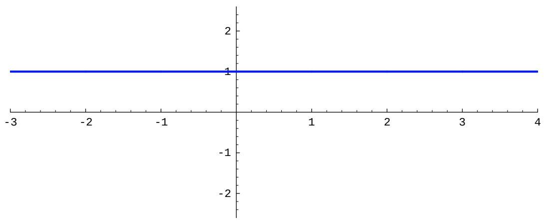
---

3.4.1. (a) $$\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sin k\pi x}{k} - \frac{8}{\pi^3}\sum_{j=0}^{\infty}\frac{\sin(2j+1)\pi x}{(2j+1)^3};\qquad (b)\ \ \frac{1}{3}+\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k\cos k\pi x}{k^2}.$$

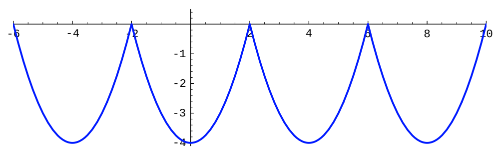
3.4.2. (a) Sine series: $\dfrac{4}{\pi}\displaystyle\sum_{j=0}^{\infty}\frac{\sin(2j+1)\pi x}{2j+1}$;




cosine series: 1;




3.4.3. (b) $-\dfrac{8}{3}+\dfrac{16}{\pi^2}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\frac{k\pi x}{2}$;




3.4.4. The differentiated Fourier series only converges when the periodic extension of the function is continuous: (b) $\dfrac{8}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin\frac{k\pi x}{2}$: converges to the 4–periodic extension of $2x$.

3.4.5. (b) $\dfrac{x^3}{3}-4x \sim -\dfrac{8}{3}x+\dfrac{32}{\pi^3}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\sin\frac{k\pi x}{2} \sim \frac{32}{3\pi}\sum_{k=1}^{\infty}\left(\frac{\pi^2k^2+3}{\pi^2k^3}\right)\sin\frac{k\pi x}{2}$.

---

3.5.2. (a) converges to $\left(0,\frac{1}{2}\right)$; (c) converges to $(0,0)$; (e) converges to $(0,0)$.

3.5.3. (a) Converges pointwise to the constant function 1;

(c) converges pointwise to the function $f(x)=\begin{cases}1, & x=0,\\ 0, & x\neq 0.\end{cases}$

3.5.5. (a) Pointwise, but not uniformly; (c) both.

3.5.6. It converges pointwise since, for each $x\neq 0$, as $n\to\infty$, the exponential term goes to zero faster than the linear term in $n$; on the other hand, $f_n(0)=0$ for all $n$. It does not converge uniformly since $\max v_n=\sqrt{n/(2e)}\not\to 0$.

3.5.7. (b) pointwise.

3.5.11. (a) Uniformly convergent; (c) doesn't pass test.

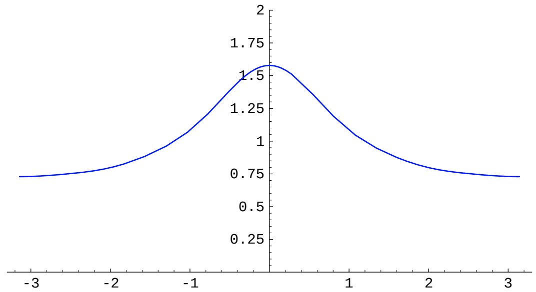
♢ 3.5.15. According to (3.66), $|\,c_k\,|=\frac{1}{2}\sqrt{a_k^2+b_k^2}$, and hence the condition (3.97) holds. Thus, the result follows immediately from Theorem 3.29.

---

3.5.21. (a) The periodic extension is not continuous, and so the best one could hope for is $a_k,b_k\to 0$ like $1/k$. Indeed, $a_0=-2\pi$, $a_k=0$, $b_k=(-1)^{k+1}2/k$, for $k>0$.

3.5.22. (*b*) $\mathrm{C}^3$; (*d*) not even continuous.

♣ 3.5.23. (*a*) This sums to a smooth, $\mathrm{C}^\infty$ function.




The error in the $n^{\text{th}}$ partial sum is bounded by $\displaystyle\sum_{k=n+1}^{\infty} e^{-k} = \frac{e^{-n}}{e-1}$ which is $\approx .0039$ when $n = 5$, and so summing from $k = 0$ to $5$ will produce accuracy in the second decimal place on the entire interval.

---

3.5.26. (*a*) Converges in norm.

3.5.27. (*b*) Converges pointwise to $x$; does not converge in $\mathrm{L}^2$ norm.

3.5.31. In 3.5.22: (*b*) $\displaystyle\sqrt{\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{1}{k^4(1+k^3)^2}}\,$; (*d*) $\displaystyle\sqrt{\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}}\,$.

♢ 3.5.34. (*a*)
$$\begin{aligned}\|\, f+g \,\|^2 &= \langle\, f+g \,;f+g \,\rangle = \|\, f \,\|^2 + \langle\, f \,;g \,\rangle + \langle\, g \,;f \,\rangle + \|\, g \,\|^2 \\ &= \|\, f \,\|^2 + \langle\, f \,;g \,\rangle + \overline{\langle\, f \,;g \,\rangle} + \|\, g \,\|^2 = \|\, f \,\|^2 + 2\,\mathrm{Re}\,\langle\, f \,;g \,\rangle + \|\, g \,\|^2.\end{aligned}$$

3.5.37. (*a*) The complex Fourier coefficients of $f(x) = x$ are $c_k = \begin{cases} (-1)^k\,\mathrm{i}\,/k, & k \neq 0, \\ 0, & k = 0. \end{cases}$

Thus, Plancherel's formula is
$$\frac{\pi^3}{3} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{k=-\infty}^{\infty} |\,c_k\,|^2 = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{1}{k^2} = 2\sum_{k=1}^{\infty}\frac{1}{k^2},$$
which coincides with (3.57).

♢ 3.5.41. Note first that, for $1 \leq k \leq n$,
$$0 \leq |\,v_i^{(n)} - v_i^\star\,| \leq \sqrt{\left(v_1^{(n)} - v_1^\star\right)^2 + \;\cdots\; + \left(v_m^{(n)} - v_m^\star\right)^2} = \|\,\mathbf{v}^{(n)} - \mathbf{v}^\star\,\|,$$
and hence if $\|\,\mathbf{v}^{(n)} - \mathbf{v}^\star\,\| \to 0$, then $|\,v_i^{(n)} - v_i^\star\,| \to 0$ and so $v_i^{(n)} \to v_i^\star$.

On the other hand, if $v_i^{(n)} \to v_i^\star$ for all $i = 1, \ldots, m$, then
$$\|\,\mathbf{v}^{(n)} - \mathbf{v}^\star\,\| = \sqrt{\left(v_1^{(n)} - v_1^\star\right)^2 + \;\cdots\; + \left(v_n^{(m)} - v_m^\star\right)^2} \quad\longrightarrow\quad 0. \qquad\qquad \textit{Q.E.D.}$$

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# Student Solutions to

## Chapter 4: Separation of Variables

4.1.1. (a) $u(t,x) \to u_\star(x) = 10\,x$; (b) for most initial conditions, at the exponential rate $e^{-\pi^2 t}$; others have faster decay rate; (c) for the same initial conditions as in part (b), when $t \gg 0$, the temperature $u(t,x) \approx 10\,x + c\,e^{-\pi^2 t}\sin \pi x$ for some $c \neq 0$ .

4.1.4. The solution is

$$u(t,x) = \sum_{n=1}^{\infty} d_n \,\exp\left[ -\left(n+\tfrac{1}{2}\right)^2 \pi^2 t\right] \,\sin\left(n+\tfrac{1}{2}\right)\pi x$$

where

$$d_n = 2\int_0^1 f(x)\sin\left(n+\tfrac{1}{2}\right)\pi x\,dx$$

are the "mixed" Fourier coefficients of the initial temperature $u(0,x) = f(x)$. All solutions decay exponentially fast to zero: $u(t,x) \to 0$ as $t \to \infty$. For most initial conditions, i.e., those for which $d_1 \neq 0$, the decay rate is $e^{-\pi^2 t/4} \approx e^{-2.4674\,t}$. The solution profile eventually looks like a rapidly decaying version of the first eigenmode $\sin \frac{1}{2}\pi x$.

4.1.10. (a) $u(t,x) = e^{-t}\cos x$; equilibrium temperature: $u(t,x) \to 0$.

♢ 4.1.13. Since $u(t,x) \to 0$ uniformly in $x$, the thermal energy $E(t) = \displaystyle\int_0^\ell u(t,x)\,dx \to 0$ also. So if $E(t_0) \neq 0$, then $E(t)$ cannot be constant. On physical grounds, the energy is not constant due to the nonzero heat flux through the ends of the bar, as measured by the boundary terms in

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^\ell u(t,x)\,dx = \int_0^\ell \frac{\partial u}{\partial t}(t,x)\,dx = \int_0^\ell \frac{\partial^2 u}{\partial x^2}(t,x)\,dx = \frac{\partial u}{\partial x}(t,\ell) - \frac{\partial u}{\partial x}(t,0).$$

Thus, in general, $E'(t) \neq 0$, which implies that $E(t)$ is not constant.

♢ 4.1.17. By the chain rule, $v_t = u_t + c\,u_x = \gamma\,u_{xx} = \gamma\,v_{xx}$. The change of variables represents a Galilean boost to a coordinate system that is moving with the fluid at speed $c$.

---

4.2.3. (b) $u(t,x) = \dfrac{4}{\pi}\displaystyle\sum_{j=0}^{\infty} \frac{\sin\sqrt{2}\,(2j+1)t\,\sin(2j+1)x}{\sqrt{2}\,(2j+1)^2}$; (f) $u(t,x) = t-1$.

4.2.4. (b) $1,\ t,\ \cos n t \cos n x,\ \sin n t\cos n x,$ for $n = 0,1,2,\ldots$ .

♡ 4.2.9. (a) The solution formulae depend on the size of $a$. For $k = 1,2,3,\ldots$, the separable solutions are

$$\begin{array}{l} e^{-\alpha_k^+ t}\sin k\pi x, \\ e^{-\alpha_k^- t}\sin k\pi x, \end{array} \quad \text{where} \quad \alpha_k^{\pm} = \frac{a \pm \sqrt{a^2 - 4k^2\pi^2c^2}}{2}, \quad \text{for} \quad k < \frac{a}{2\pi c},$$

and, possibly,

$$e^{-a t/2}\sin k\pi x, \quad t\,e^{-a t/2}\sin k\pi x, \quad \text{provided} \quad 0 < k = \frac{a}{2\pi c} \quad \text{is an integer,}$$

and

$$e^{-a t/2} \cos \omega_k t \, \sin k \pi x, \qquad \text{where} \qquad \omega_k = \tfrac{1}{2}\sqrt{4k^2\pi^2c^2 - a^2}\,, \quad \text{for} \quad k > \frac{a}{2\pi c}\,.$$
$$e^{-a t/2} \sin \omega_k t \, \sin k \pi x,$$

In particular, if $a < 2\pi c$, then only the latter modes appear.

(*b*) For the given initial data, the series solution is

$$\sum_{k<a/(2\pi c)} b_k \,\frac{\alpha_k^+ \, e^{-\alpha_k^- t} - \alpha_k^- \, e^{-\alpha_k^+ t}}{\alpha_k^+ - \alpha_k^-} \,\sin k\pi x + \sum_{k=a/(2\pi c)} b_k e^{-a t/2}\bigl(1 + \tfrac{1}{2}\, a t\bigr)\, \sin k \pi x$$
$$+ \sum_{k>a/(2\pi c)} b_k e^{-a t/2}\left(\cos\omega_k t + \frac{a}{2\omega_k}\sin\omega_k t\right)\sin k\pi x,$$

where $k = 1, 2, 3, \ldots$ must be a positive integer, with the convention that the sum is zero if no positive integer satisfies the indicated inequality or equality, while $b_k = 2\int_0^1 f(x)\sin k\pi x\,dx$ are the usual Fourier sine coefficients of $f(x)$ on $[0,1]$.

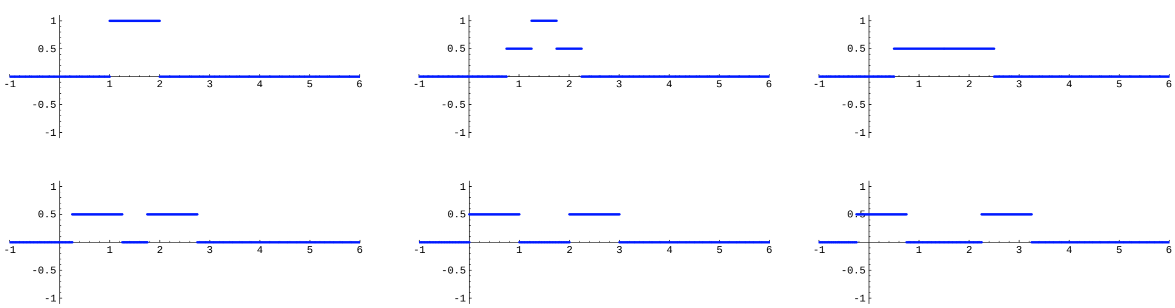
(*c*) For underdamped or critically damped motion, where $0 < a \le 2\pi c$, the modes all decay exponentially, as a rate $e^{-a t/2}$. In the overdamped case, $a > 2\pi c$, the slowest decaying mode has decay rate $e^{-\alpha_1^- t}$ where $\alpha_1^- = \dfrac{a - \sqrt{a^2 - 4\pi^2 c^2}}{2}$.

(*d*) If $a < 2\pi c$, the system is underdamped, while if $a > 2\pi c$ it is overdamped.

---

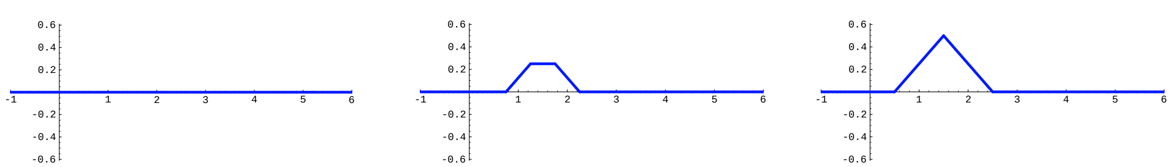
4.2.14. (a) The initial displacement splits into two half sized replicas, moving off to the right and to the left with unit speed.

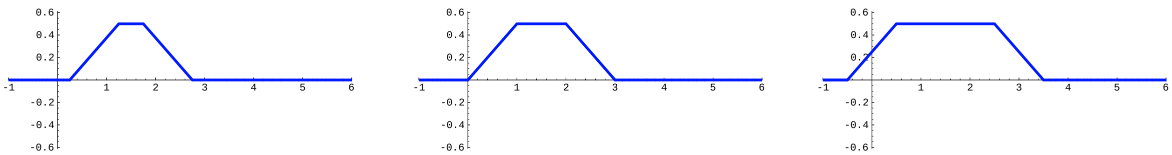
Plotted at times $t = 0, .25, .5, .75, 1., 1.25$:



4.2.15. (a) The solution initially forms a trapezoidal displacement, with linearly growing height and sides of slope $\pm 1$ expanding in both directions at unit speed, starting from $x = 1$ and 2. When the height reaches .5, it momentarily forms a triangle. Afterwards, it takes the form of an expanding trapezoidal form of fixed height .5, with the diagonal sides propagating to the right and to the left with unit speed.

Plotted at times $t = 0, .25, .5, .75, 1., 1.5$:





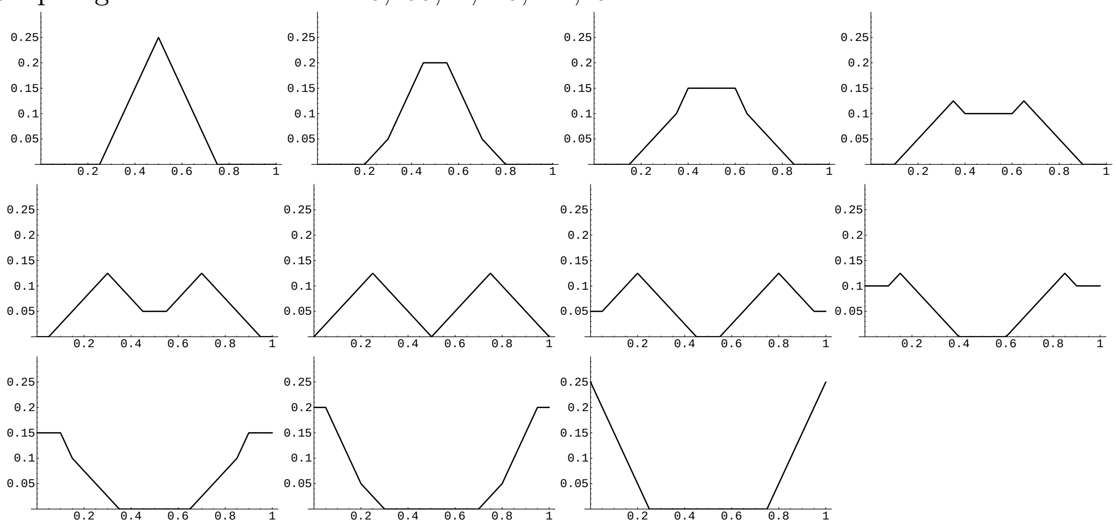

♢ 4.2.22. The solution is periodic if and only if the initial velocity has mean zero: $\displaystyle\int_0^\ell g(x)\,dx = 0$. For generic solutions, the period is $2\ell/c$, although some special solutions oscillate more rapidly.

4.2.24. (a) The initial position $f(x)$ and velocity $g(x)$ should be extended to be even functions with period 2. Then the d'Alembert formula

$$u(t,x) = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz$$

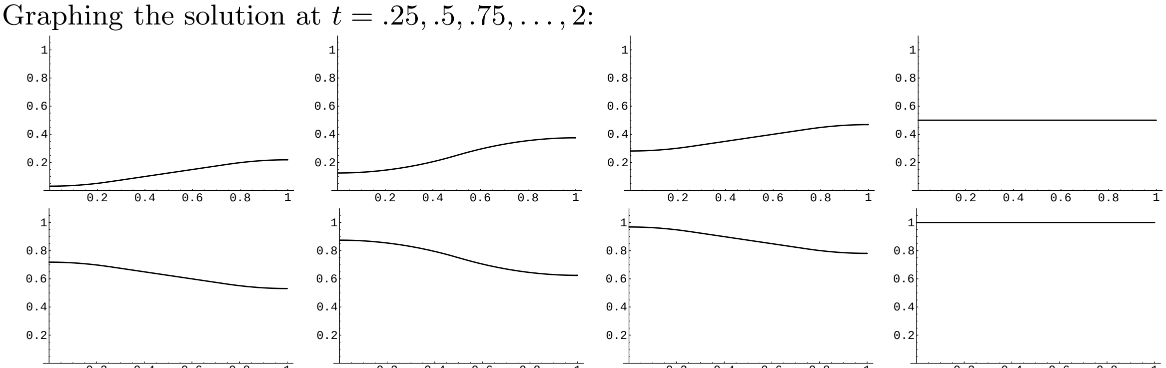
will give the solution on $0 \le x \le 1$.

($b$) Graphing the solution at $t = 0, .05, .1, .15, \ldots, .5$:




At $t = .5, \ldots, 1$, the solution has the same graphs, run in reverse order, since $u(1-t,x) = u(t,x)$. At $t = 1$ the solution repeats: $u(t+1,x) = u(t,x)$, since it is periodic of period 1.

($c$) Graphing the solution at $t = .25, .5, .75, \ldots, 2$:




After this, the solution repeats, but with an overall increase in height of 1 after each time period of 2. Indeed, $u(t+1,x) = \frac{1}{2} + u(t,-x)$, while $u(t+2,x) = u(t,x)+1$. The solution is not periodic.

♢ 4.2.28. The d'Alembert formula (4.77) implies that the solution is

$$u(t,x) = \frac{f(x-c\,t)+f(x+c\,t)}{2} + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} g(z)\,dz,$$

where $f(x)$ and $g(x)$ denote the $\ell$–periodic extensions of the initial data functions. Thus, by periodicity of the initial data and Lemma 3.19,

$$\begin{aligned} u\left(t+\frac{\ell}{c},x\right) - u(t,x) &= \frac{f(x-c\,t-\ell)-f(x-c\,t)+f(x+c\,t+\ell)-f(x+c\,t)}{2} \\ &\qquad + \frac{1}{2\,c}\left(\int_{x-c\,t-\ell}^{x+c\,t+\ell} g(z)\,dz - \int_{x-c\,t}^{x+c\,t} g(z)\,dz\right) \\ &= \frac{1}{2\,c}\left(\int_{x+c\,t}^{x+c\,t+\ell} g(z)\,dz + \int_{x-c\,t-\ell}^{x-c\,t} g(z)\,dz\right) = \frac{1}{c}\int_0^{\ell} g(z)\,dz. \end{aligned}$$

If $\displaystyle\int_0^{\ell} g(x)\,dx = 0$, then $u(t,x)$ is a periodic function of $t$ with period $\dfrac{\ell}{c}$. On the other hand, if $\displaystyle\int_0^{\ell} g(x)\,dx \neq 0$, then $u(t,x)$ increases (or decreases) by a fixed nonzero amount after each time interval of duration $\ell/c$, and so cannot be a periodic function of $t$.

♢ 4.2.31. (a) Oddness require $f(0) = -f(-0) = -f(0)$, so $f(0) = 0$. Also, $f(-\ell) = -f(\ell)$, while $f(-\ell) = f(2\ell-\ell) = f(\ell)$ by periodicity; thus $f(\ell) = 0$.

4.2.34. (a) If $u(t,x)$ is even in $t$, then $u_t(t,x)$ is odd, and so $u_t(0,x) = 0$. Vice versa, if $u_t(0,x) = g(x) = 0$, then, by the d'Alembert formula (4.77),

$$u(-t,x) = \frac{f(x+c\,t)+f(x-c\,t)}{2} = u(t,x).$$

---

4.3.2. If the force has magnitude $f < 0$,

$$-\Delta u = f, \quad x^2+y^2<1, \qquad u = 0, \quad x^2+y^2=1, \quad y>0, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0, \quad x^2+y^2=1, \quad y<0.$$

4.3.6. $\Delta u = 0, \quad u(0,y) = u(x,0) = 1, \quad u(1,y) = 1+y, \quad u(x,1) = 1+x, \quad 0<x,y<1.$

---

4.3.10. (b) $u(x,y) = \dfrac{\sinh(\pi-x)\ \sin y}{\sinh \pi}$.

4.3.12. (a) $u(x,y) = \dfrac{\sin \pi x\ \sinh \pi(1-y) + \sinh \pi(1-x)\ \sin \pi y}{\sinh p}$

4.3.13. (b) $u(x,y) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\cos n x\ \cosh n(\pi-y)}{(4n^2-1)\cosh n\,\pi}$.

4.3.17. $u(x,y) = \displaystyle\sum_{n=1}^{\infty} \frac{b_n e^{1-y} \sin n\pi x\ \sinh\sqrt{n^2\pi^2+1}\,y}{\sinh\sqrt{n^2\pi^2+1}}$, where $b_n = 2\displaystyle\int_0^1 f(x)\sin n\pi x\,dx$ are the Fourier sine coefficients of the boundary data.

---

4.3.22. (a) $u(x) = c_1 x + \dfrac{c_2}{x^5}$, (c) $u(x) = c_1 |\,x\,|^{(1+\sqrt5)/2} + c_2 |\,x\,|^{(1-\sqrt5)/2}$.

4.3.25. (a) $u(x,y) = \frac{1}{4} r^3 \cos 3\theta + \frac{3}{4} r \cos\theta = \frac{1}{4} x^3 - \frac{3}{4} x y^2 + \frac{3}{4} x.$

♡ 4.3.27. (b) $u(x,y) = 1 - \dfrac{2}{\pi} \tan^{-1}\left( \dfrac{1 - x^2 - y^2}{2y} \right), \qquad x^2 + y^2 < 1, \qquad y > 0.$

4.3.31. Since the boundary conditions are radially symmetric, $u$ must also be radially symmetric, and hence a linear combination of $\log r$ and 1. A short computation shows that

$$u = \frac{b-a}{\log 2} \log r + b = \frac{b-a}{2\log 2} \log(x^2 + y^2) + b.$$

4.3.34. (b) $u(r,\theta) = \dfrac{2}{3}\left( r - \dfrac{1}{r} \right) \cos\theta, \qquad$ (f) no solution.

♢ 4.3.40. First,

$$\int \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}\, d\phi = \tan^{-1}\left( \frac{1+r}{1-r} \tan\frac{\phi - \theta}{2} \right).$$

To evaluate the definite integral, from $\phi = 0$ to $\pi$, we need to be careful about the branches of the inverse tangent:

$$u(r,\theta) = \frac{1}{2\pi} \int_0^\pi \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}\, d\phi = \begin{cases} 1 - \dfrac{1}{\pi} \tan^{-1}\left( \dfrac{1 - r^2}{2r\sin\theta} \right), & 0 < \theta < \pi, \\[2ex] \frac{1}{2}, & \theta = 0, \pm\pi, \\[2ex] -\dfrac{1}{\pi} \tan^{-1}\left( \dfrac{1 - r^2}{2r\sin\theta} \right), & -\pi < \theta < 0. \end{cases}$$

where we use the usual branch $-\frac{1}{2}\pi < \tan^{-1} t < \frac{1}{2}\pi$ of the inverse tangent.

4.3.44. For example, $u(x,y) = 1 - x^2 - y^2$ satisfies $-\Delta u = 4$, and achieves its maximum at $x = y = 0$. It represents the displacement of a circular membrane due to a uniform upwards force of magnitude 4.

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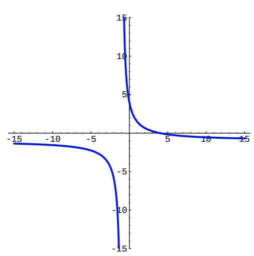
4.4.1. (b) hyperbola:




4.4.2. (a) Elliptic; (f) hyperbolic.

4.4.3. Elliptic when $x(t+x) > 0$; parabolic when $t = -x$ or $x = 0$, but not both; hyperbolic when $x(t+x) < 0$; degenerate at the origin $t = x = 0$.

4.4.6. Written out, it becomes

$$L[u] = -p(x,y)\, u_{xx} - q(x,y)\, u_{yy} - p_x(x,y)\, u_x - q_y(x,y)\, u_y + r(x,y)\, u = f(x,y),$$

with discriminant $\Delta = -4p(x,y)\, q(x,y)$; hence elliptic if and only if $p(x,y)\, q(x,y) > 0$.

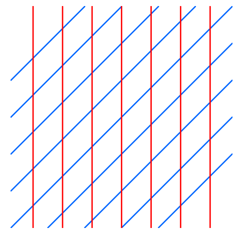
4.4.13. According to (4.139) (with $y$ replacing $t$), the discriminant is

$$\Delta = (2\, u_x u_y)^2 - 4\,(1 + u_x^2)\,(1 + u_y^2) = -4\,(1 + u_x^2 + u_y^2) < 0,$$

and hence the equation is elliptic everywhere.

4.4.14. (a) No real characteristics;

(f) vertical lines $t = a$ or lines $x = t+b$ of slope 1:



4.4.18. (a) Parabolic when $y = 0$; hyperbolic everywhere else.

(b) The characteristics satisfy the ordinary differential equation $\left(\dfrac{dy}{dx}\right)^2 - y\,\dfrac{dy}{dx} = 0$. Thus, either $\dfrac{dy}{dx} = 0$ and so $y = k$, or $\dfrac{dy}{dx} = y$ and so $y = \pm c\,e^x$.

(c) The characteristic coordinates are $\xi = y\,e^{-x}$, $\eta = y$. By the chain rule, the equation for $u = v(\xi,\eta) = v(y\,e^{-x}, y)$ becomes $-\xi\,\eta\,v_{\xi\eta} = \eta^2$, with general solution

$$v = F(\xi) + G(\eta) + \tfrac{1}{2}\,\eta^2 \log\xi, \quad \text{whence} \quad u = F(y) + G(y\,e^{-x}) + \tfrac{1}{2}\,y^2(x - \log y),$$

where $F, G$ are arbitrary scalar functions.

4.4.19. (b) $u_{xx} + 2\,y\,u_{xy} + y^2 u_{yy} = 0$; one can also include arbitrary lower-order terms.

# Student Solutions to

## Chapter 5: Finite Differences

♣ 5.1.1. (*b*) $u'(1) = -.5$; finite difference approximations: $-.475113, -.497500, -.499750$; errors: $.024887, .002500, .000250$; first-order approximation.

♣ 5.1.2. (*b*) $u'(1) = -.5$; finite difference approximations: $-.4999875, -.49999999875$, $-.49999999999986$; errors: $1.25 \times 10^{-5}, 1.25 \times 10^{-9}, 1.38 \times 10^{-13}$; looks like a fourth-order approximation.

♣ 5.1.3. (*b*) $u''(1) = .5$; finite difference approximations: $.49748756, .49997500, .49999975$; errors: $-2.51 \times 10^{-3}, -2.50 \times 10^{-5}, -2.50 \times 10^{-7}$; second-order approximation.

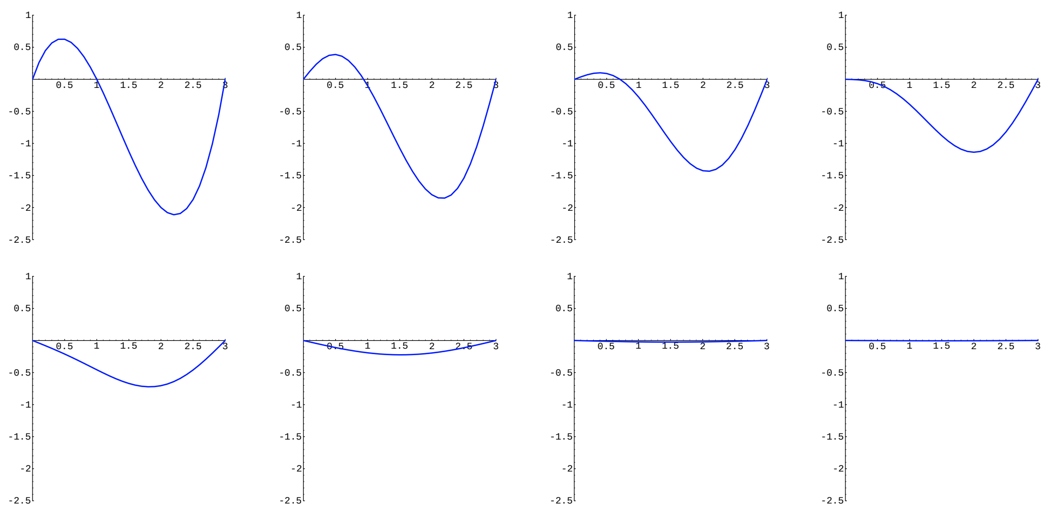
♠ 5.1.5. (a) $$u'(x) = \frac{-3\,u(x) + 4\,u(x+h) - u(x+2\,h)}{2\,h} \; + \; \mathrm{O}(h^2).$$

(*c*) The errors in computing $u'(1) = 5.43656$ are, respectively, $-2.45 \times 10^{-1}$, $-1.86 \times 10^{-3}$, $-1.82 \times 10^{-5}$, which is compatible with a second-order appproximation because each decrease in step size by $\frac{1}{10} = 10^{-1}$ decreases the error by approximately $\left(\frac{1}{10}\right)^2 = 10^{-2}$.

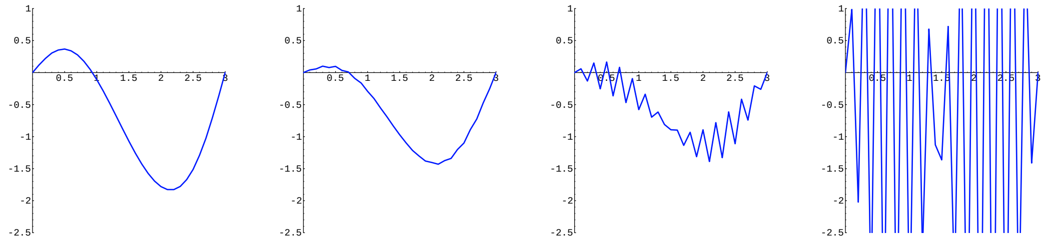
---

5.2.1. (*a*) $0 < \Delta t \leq .001$.

(*b*) For $\Delta t = .001$:, we plot the numerical solution at times $t = 0., .01, .03, .05, .1, .3, .5, 1.$:

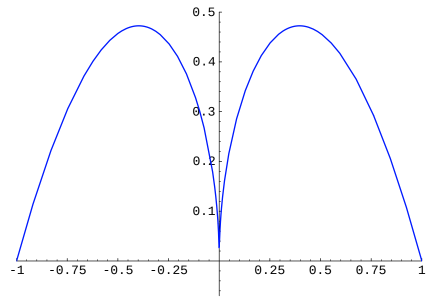


For $\Delta t = .0011$:, we plot the numerical solution at times $t = .011, .0308, .0506, .0704$:




The former is a good approximation, whereas the latter is clearly unstable.

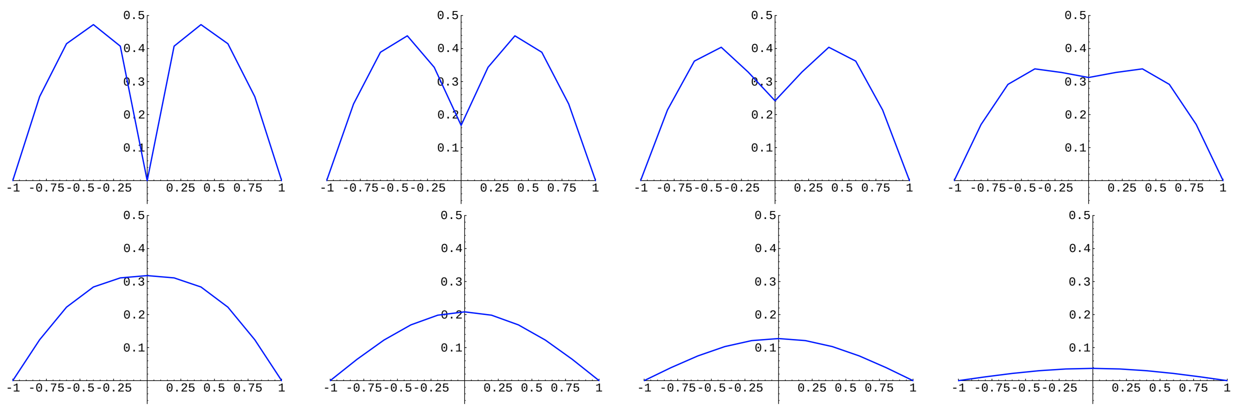
5.2.4. Before approximation, the initial data is



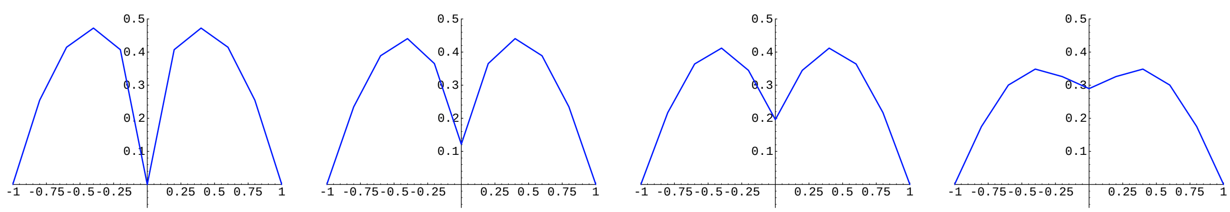

In each case, we graph the numerical solution using piecewise affine interpolation between data points.

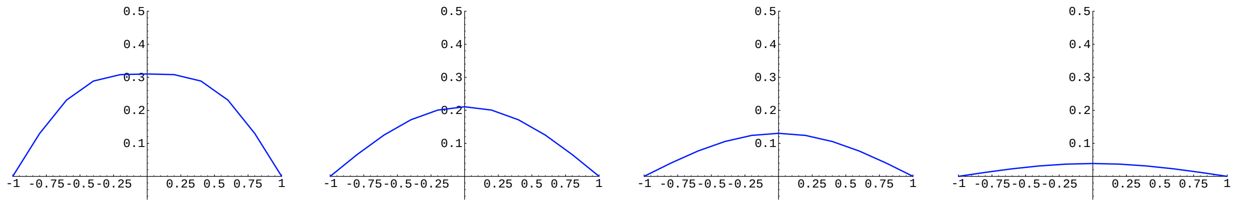
(a) For $\Delta x = .1$:

($i$) With $\Delta t = .005$, so that $\mu = .5$:

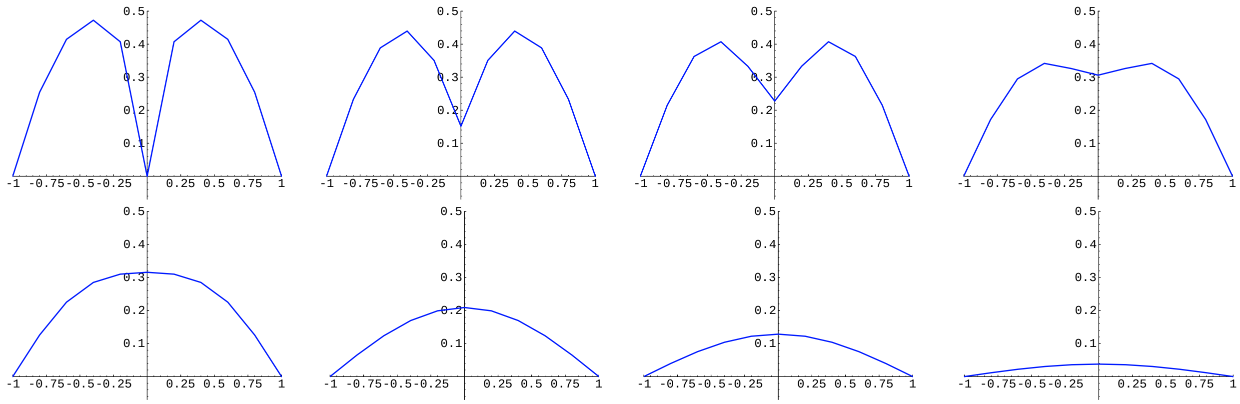



($ii$) With $\Delta t = .01$:





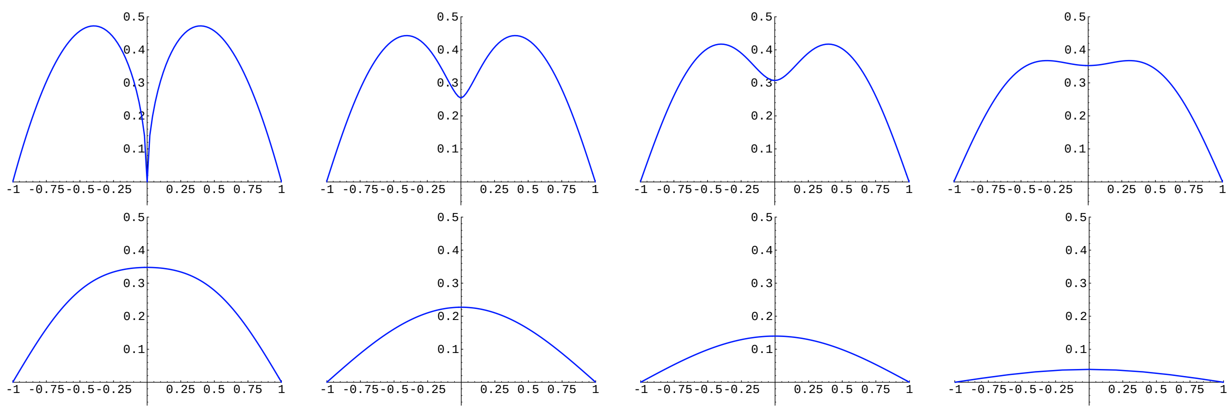

$(iii)$ With $\Delta t = .01$:



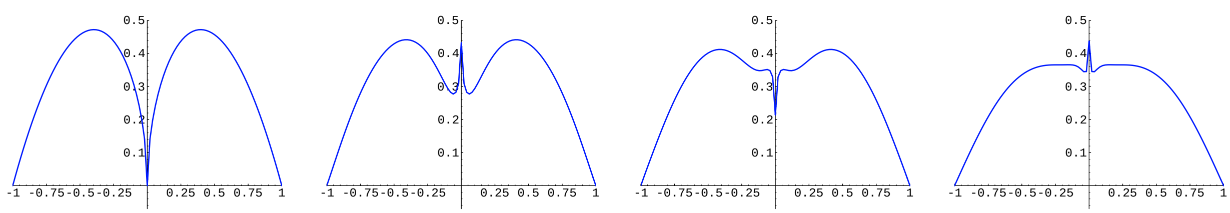

$(b)$ For $\Delta x = .01$:

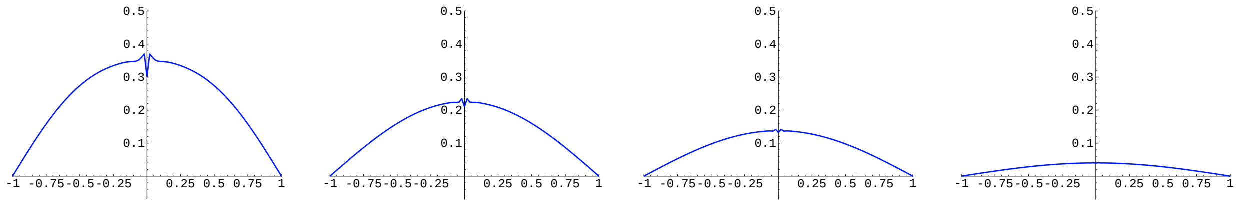
$(i)$ We would need $\Delta t < .00001$, which requires too much computation!

$(ii)$ With $\Delta t = .01$:




$(iii)$ With $\Delta t = .01$:






Note the undesirable oscillations in the Crank-Nicolson scheme due to the singularity in the initial data at $x = 0$.

5.2.8. (a) Set $\Delta x = 1/n$. The approximations $u_{j,m} \approx u(t_j, x_m) = u(j\,\Delta t, m\,\Delta x)$ are iteratively computed using the explicit scheme

$$u_{j+1,m} = \mu\, u_{j,m+1} + (1 - 2\,\mu - \alpha\,\Delta t) u_{j,m} + \mu\, u_{j,m-1}, \qquad \begin{array}{l} j = 0, 1, 2, \ldots\,, \\ m = 1, \ldots, n-1, \end{array}$$

where $\mu = \Delta t/(\Delta x)^2$, along with boundary conditions $u_{j,0} = u_{j,n} = 0$ and initial conditions $u_{0,m} = f_m = f(x_m)$.

($b$) Applying the von Neumann stability analysis, the magnification factor is

$$\lambda = 1 - 4\,\mu \sin^2\left(\tfrac{1}{2}\,k\,\Delta x\right) - \alpha\,\Delta t.$$

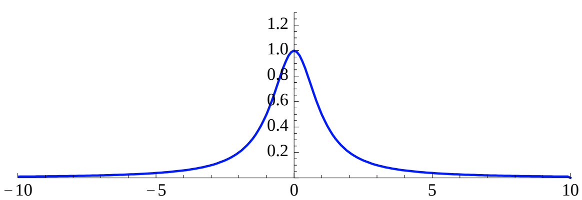
Stability requires $0 \leq 4\,\mu + \alpha\,\Delta t \leq 2$, and hence, since we are assuming $\alpha > 0$,

$$\Delta t \leq \frac{(\Delta x)^2}{2 + \frac{1}{2}\,\alpha\,(\Delta x)^2} \approx \tfrac{1}{2}\,(\Delta x)^2,$$

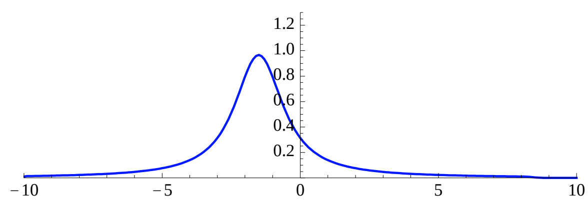
for sufficiently small $\Delta x \ll 1$.

---

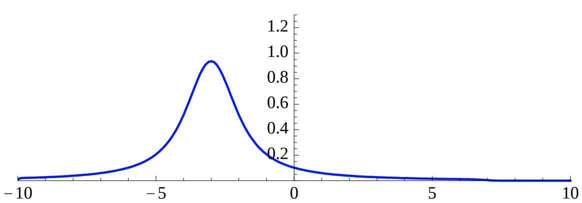
5.3.1. We set $\Delta t = .03$ to satisfy the CFL condition (5.41):



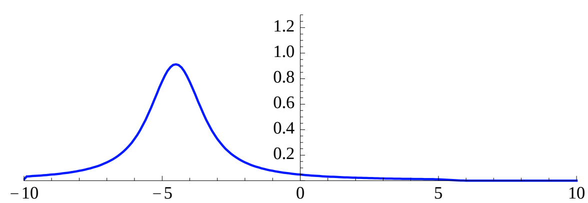

$t = 0$




$t = .5$

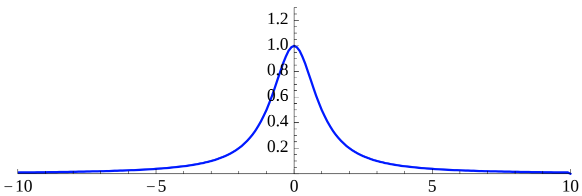



$t = 1$

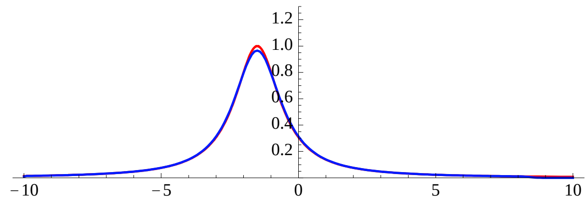



$t = 1.5$

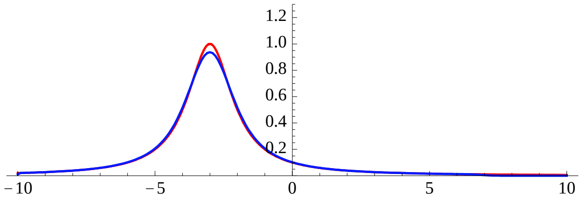
The solution is reasonably accurate, showing the wave moving to the left with speed $c = -3$. Comparing the numerical solution with the explicit solution $u(t, x) = f(x + 3t)$ at the same times, we see that the numerical solution loses amplitude as it evolves:



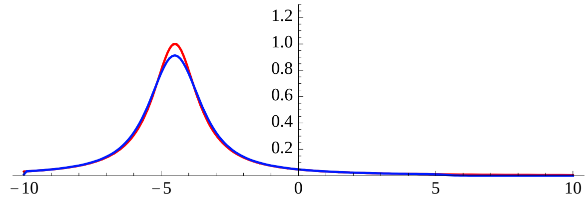

$t = 0$




$t = .5$



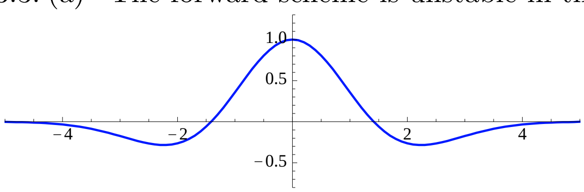

$t = 1$



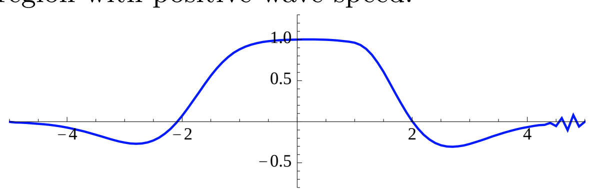

$t = 1.5$

At the three times, the maximum discrepancies ($\mathrm{L}^\infty$ norm of the difference between the exact and numerical solutions) are, respectively, $.0340, 0.0625, 0.0872$.

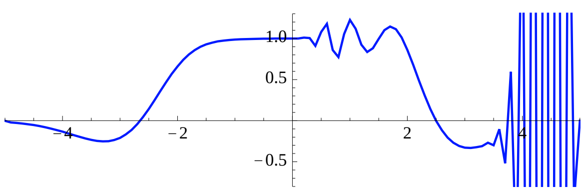
5.3.3. (*a*) The forward scheme is unstable in the region with positive wave speed:



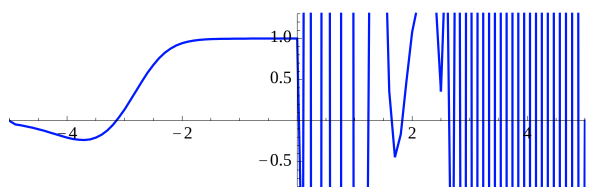

$t = 0$




$t = .5$

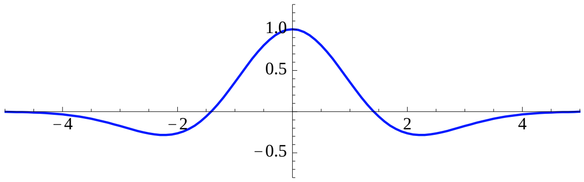



$t = 1$

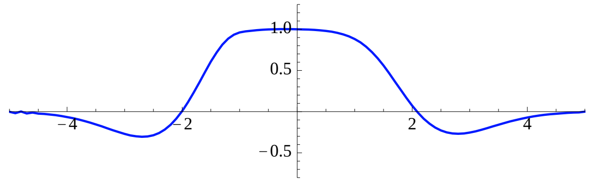



$t = 1.5$

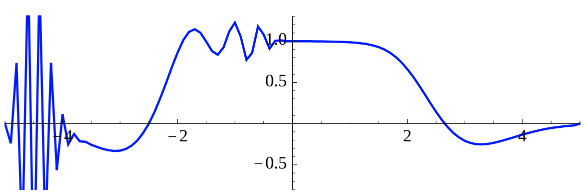
(*b*) The backward scheme is unstable in the region with negative wave speed:



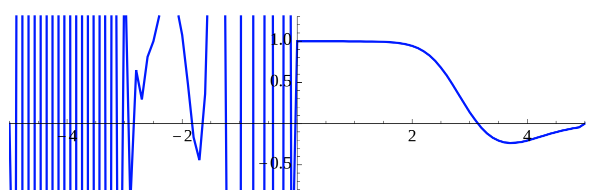

$t = 0$




$t = .5$

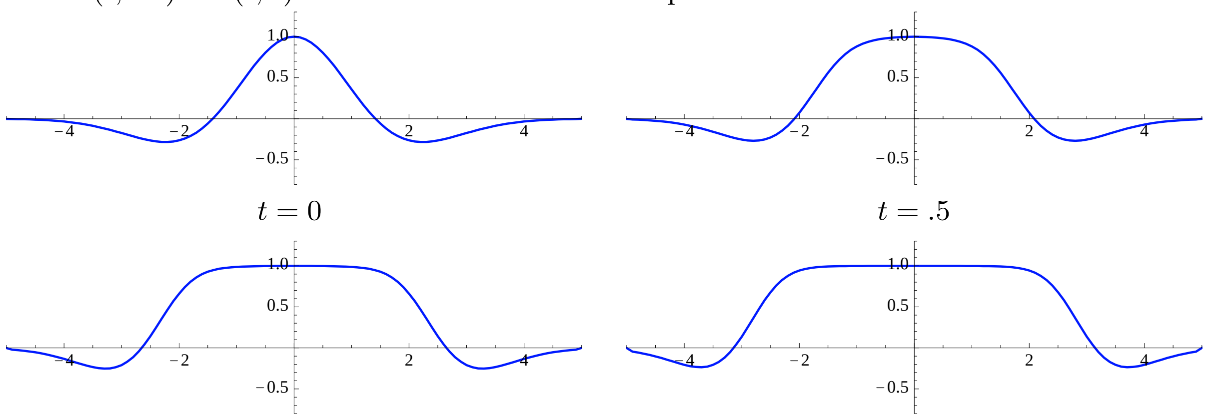



$t = 1$




$t = 1.5$

($c$) The upwind scheme is stable in both regions, and produces a reasonably accurate approximation to the solution. However, a small effect due to the boundary conditions $u(t,-5) = u(t,5) = 0$ can be seen in the final plot.



$t = 0$ $\qquad$ $t = .5$

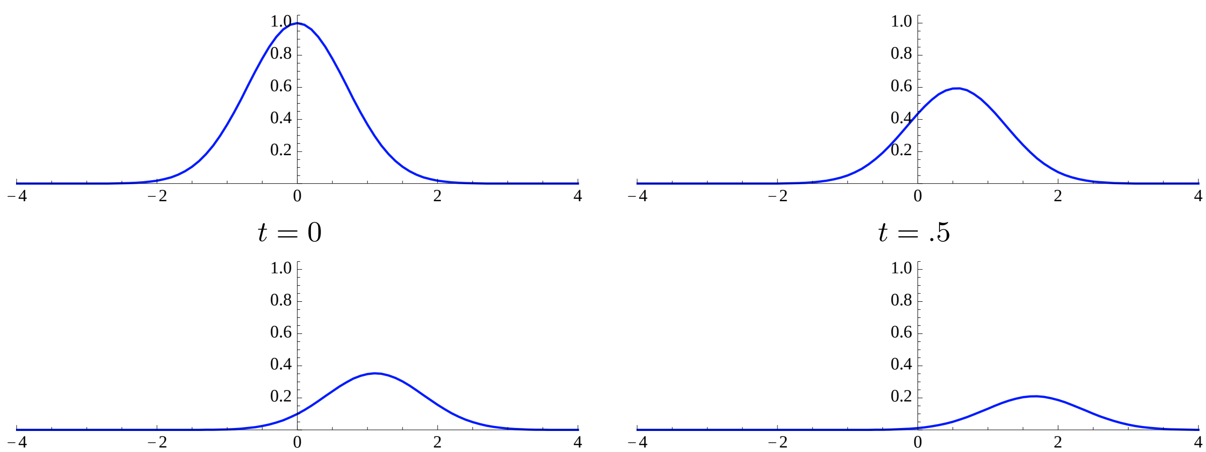
$t = 1$ $\qquad$ $t = 1.5$

5.3.8. ($a$) Since $c > 0$, we adapt the backwards scheme (5.44), leading to the iterative step

$$u_{j+1,m} = (1 - \sigma - \Delta t)u_{j,m} + \sigma\, u_{j,m-1},$$

subject to the boundary condition $u(t,a) = 0$.

($b$) We choose $\Delta x = \Delta t = .01$ and work on the interval $-4 \le x \le 4$. The resulting n umerical solution gives a reasonably good approximation to the actual solution: a damped wave moving with wave speed $c = .75$.

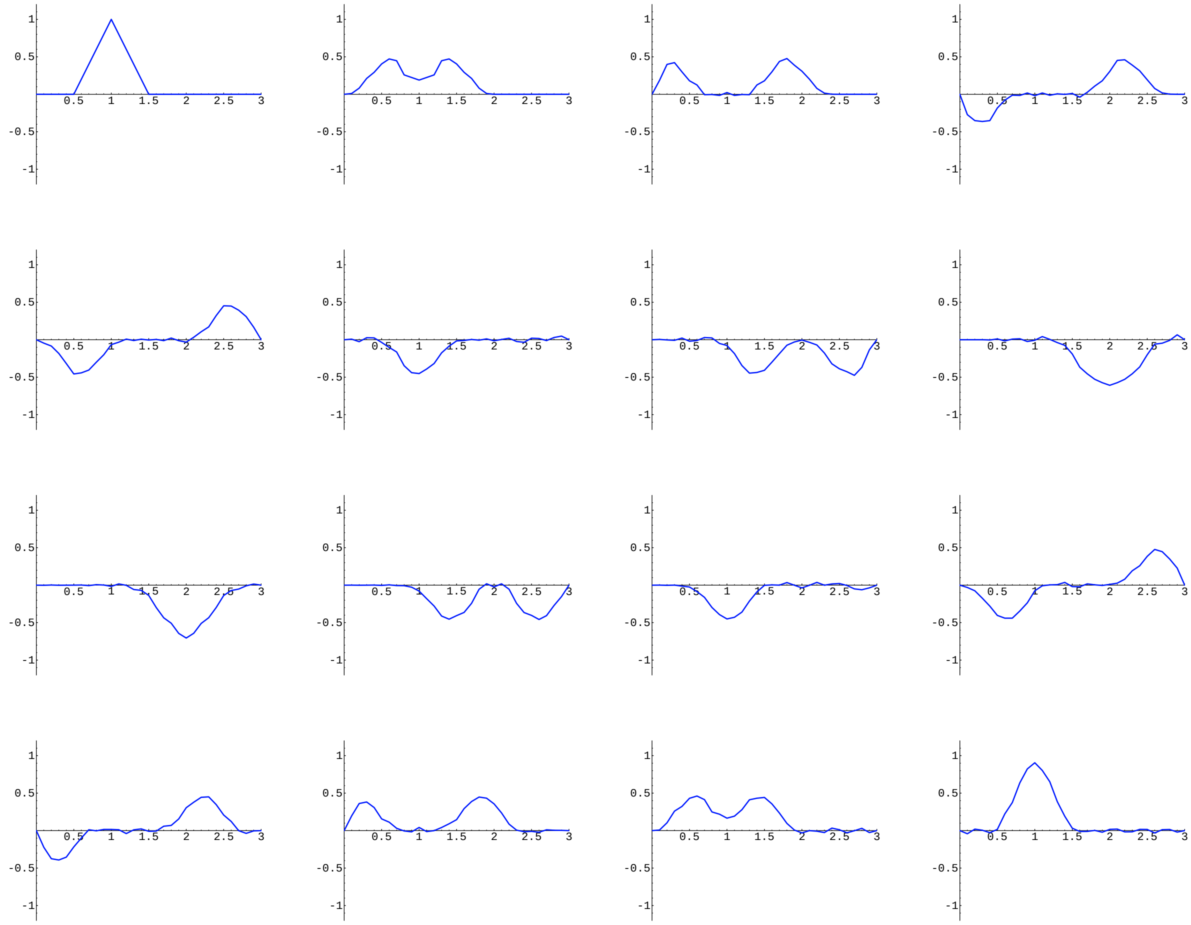


$t = 0$ $\qquad$ $t = .5$

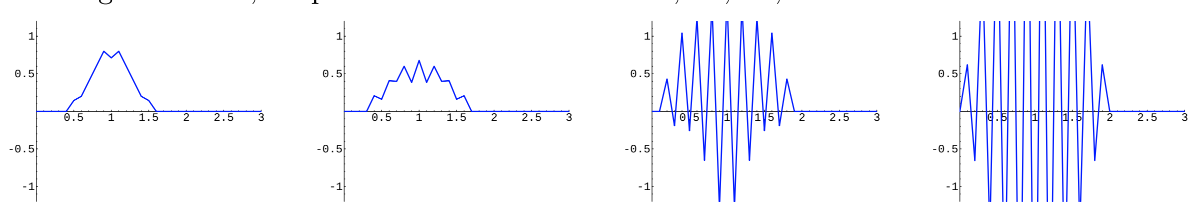
$t = 1$ $\qquad$ $t = 1.5$

5.4.1. (a) For $\Delta x = .1$, we must have $0 < \Delta t < .0125$.

(b) Setting $\Delta t = .01$, we plot the solution at times $t = 0, .05, .1, .15, .2, \ldots, .75$, which is the time at which the analytical solution repeats periodically:

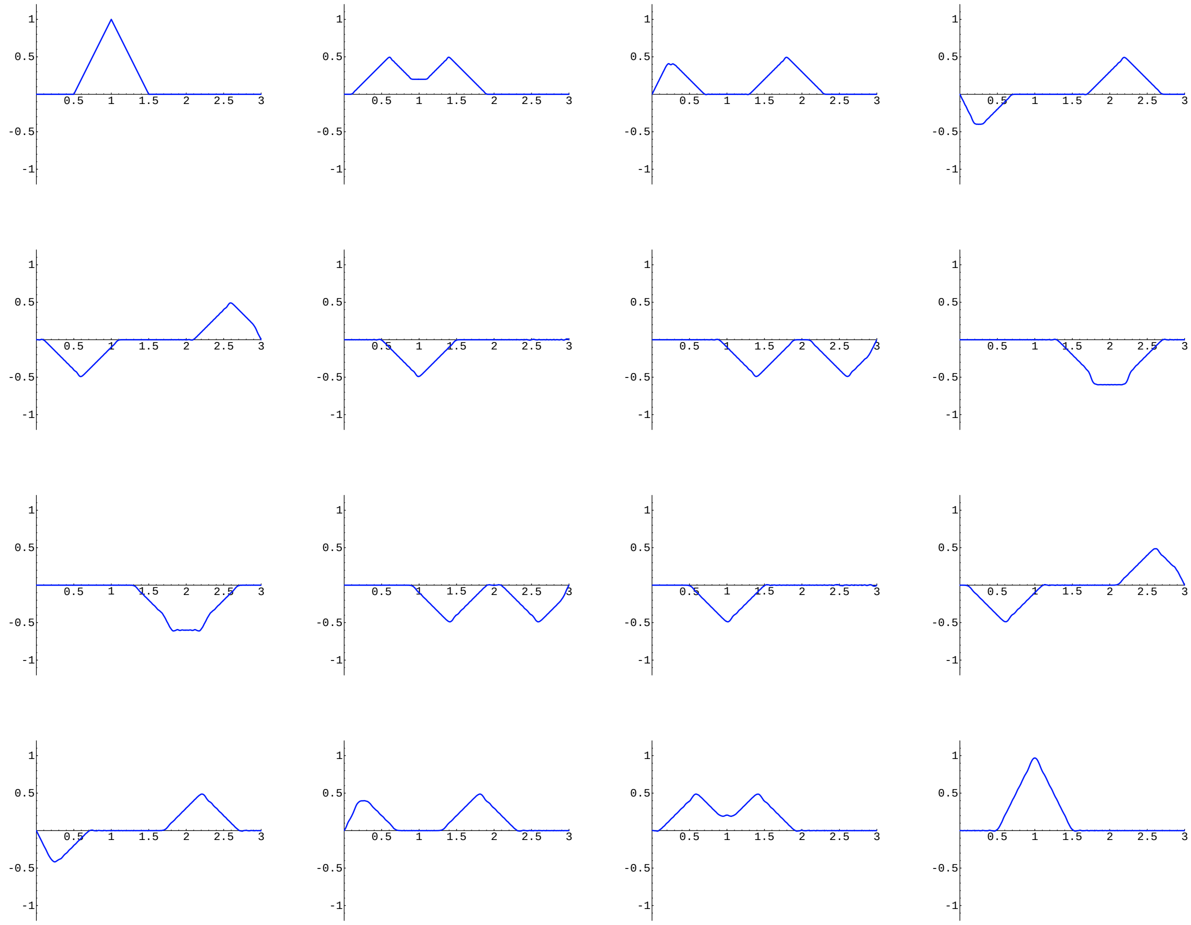


The solution has the basic features correct, but clearly is not particularly accurate. Setting $\Delta t = .015$, we plot the solution at $t = .015, .03, .06, .075$:



which is clearly unstable.

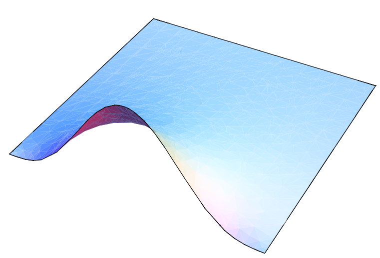
(c) For $\Delta x = .01$, we choose $\Delta t = .001$, leading to a considerably more accurate solution, again plotted at $t = 0, .05, .1, \ldots, .75$:



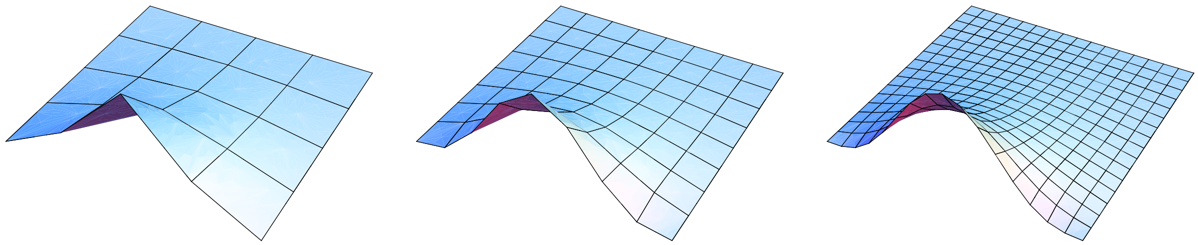
♠ 5.5.1. According to Exercise 4.3.10(a), the exact solution is

$$u(x,y) = \frac{3 \sin x \ \sinh(\pi - y)}{4 \sinh \pi} - \frac{\sin 3x \ \sinh 3(\pi - y)}{4 \sinh 3\pi},$$

with graph

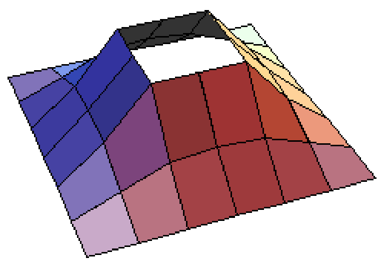


Plotting the finite difference approximations based on $n = 4, 8$ and 16 mesh points:



The maximal absolute errors between the approximations and the exact solution on the mesh points are, respectively, $.03659, .01202, .003174$. Each reduction in mesh size by a factor of $\frac{1}{2}$ leads to an reduction in the error by approximately $\frac{1}{4}$, indicative of a second order scheme.

♣ 5.5.6. (*a*) At the 5 interior nodes on each side of the central square $C$, the computed temperatures are $20.8333, 41.6667, 45.8333, 41.6667, 20.8333$:



(*b*) (*i*) The minimum temperature on $C$ is 20.8333, achieved at the four corners;
(*ii*) the maximum temperature is 45.8333, achieved at the four midpoints;
(*iii*) the temperature is not equal to $50^\circ$ anywhere on $C$.

# Student Solutions to

## Chapter 6: Generalized Functions and Green's Functions

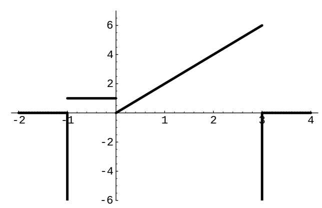
6.1.1. (a) $1$, (c) $e$.

6.1.2. (a) $\varphi(x) = \delta(x)$; $\displaystyle\int_a^b \varphi(x)\,u(x)\,dx = \begin{cases} u(0), & a < 0 < b, \\ 0, & 0 < a < b \quad \text{or} \quad a < b < 0. \end{cases}$

(c) $\varphi(x) = 3\,\delta(x-1) + 3\,\delta(x+1)$;

$$\int_a^b \varphi(x)\,u(x)\,dx = \begin{cases} 3\,u(1) + 3\,u(-1), & a < -1 < 1 < b, \\ 3\,u(1), & -1 < a < 1 < b, \\ 3\,u(-1), & a < -1 < b < 1, \\ 0, & 1 < a < b \quad \text{or} \quad -1 < a < b < 1 \\ & \qquad \text{or} \quad a < b < -1. \end{cases}$$

6.1.4. (a) $f'(x) = -\,\delta(x+1) - 9\,\delta(x-3) + \begin{cases} 2x, & 0 < x < 3, \\ 1, & -1 < x < 0, \\ 0, & \text{otherwise.} \end{cases}$




6.1.6. (b) $f'(x) = \begin{cases} -1 & x < 0, \\ 3, & 0 < x < 1, \\ 1, & x > 1, \end{cases} = -1 + 4\,\sigma(x) - 2\,\sigma(x-1), \qquad f''(x) = 4\,\delta(x) - 2\,\delta(x-1).$

6.1.11. (a) $x\,\delta(x) = \displaystyle\lim_{n\to\infty} \frac{n\,x}{\pi\,(1+n^2x^2)} = 0$ for all $x$, including $x = 0$. Moreover, the functions are all bounded in absolute value by $\frac{1}{2}$, and so the limit, although nonuniform, is to an ordinary function.

(b) $\langle\, u(x)\,;x\,\delta(x)\,\rangle = \displaystyle\int_a^b u(x)\,x\,\delta(x)\,dx = u(0)\,0 = 0$ for all continuous functions $u(x)$, and so $x\,\delta(x)$ has the same dual effect as the zero function: $\langle\, u(x)\,;0\,\rangle = 0$ for all $u$.

6.1.13. (a) When $\lambda > 0$, the product $\lambda x$ has the same sign as $x$, and so

$$\sigma(\lambda x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} = \sigma(x).$$

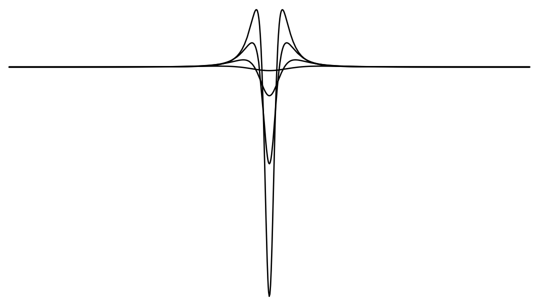
(b) If $\lambda < 0$, then $\sigma(\lambda x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} = 1 - \sigma(x).$

(c) Differentiate using the chain rule: If $\lambda > 0$, then $\delta(x) = \sigma'(x) = \lambda\,\sigma'(\lambda x) = \lambda\,\delta(\lambda x)$, while if $\lambda < 0$, then $\delta(x) = \sigma'(x) = -\,\lambda\,\sigma'(\lambda x) = -\,\lambda\,\delta(\lambda x)$. *Q.E.D.*

6.1.17. (a) Differentiating (6.32):

$$\delta''(x) = \lim_{n\to\infty} \frac{6n^5x^2 - 2n^3}{\pi(n^2x^2+1)^3}.$$

Graphs for $n = 5, 10, 15, 20$ are as follows, where the vertical range is $-5{,}500$ to $2{,}000$:



(b) $\displaystyle\int_a^b \delta''(x)(x)\,u(x)\,dx = u''(0)$, for any $u \in \mathrm{C}^2[a,b]$ and any interval with $a < 0 < b$. (If $a < b < 0$ or $0 < a < b$, the result is 0.)

♢ 6.1.21. (a) For $a < 0 < b$ and any test function $u(x)$ on $[a,b]$,

$$\langle f\,\delta'\,;u\rangle = \int_a^b u(x)\,f(x)\,\delta'(x)\,dx = -\,[u(x)\,f(x)]'\big|_{x=0} = -u'(0)\,f(0) - u(0)\,f'(0)$$
$$= \int_a^b \big[\,u(x)\,f(0)\,\delta'(x) - u(x)\,f'(0)\,\delta(x)\,\big]\,dx = \langle f(0)\,\delta' - f'(0)\,\delta\,;u\rangle.$$

6.1.22. (a) $\varphi(x) = -2\,\delta'(x) - \delta(x), \qquad \displaystyle\int_{-\infty}^{\infty} \varphi(x)\,u(x)\,dx = 2\,u'(0) - u(0).$

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6.1.27.

$$\delta(x-\xi) \;\sim\; \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{\,\mathrm{i}\,k\,(x-\xi)} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-\,\mathrm{i}\,k\,\xi}\,e^{\,\mathrm{i}\,k\,x}$$
$$\sim\; \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\big[\cos k\,\xi\,\cos k\,x + \sin k\,\xi\,\sin k\,x\big].$$

They both represent the $2\pi$–periodic extension of $\delta(x-\xi)$, namely

$$\sum_{n=-\infty}^{\infty} \delta(x-\xi-2\,n\,\pi).$$

6.1.34. (a) One way is to define it as the limit $\widetilde{\delta}(x) = \lim_{n\to\infty} G_n(x)$, where $G_n(x)$ denotes the $2\pi$–periodic extension of the function $g_n(x) = \dfrac{n}{\pi(1+n^2x^2)}$ appearing in (6.10). Alternatively, we can set $\widetilde{\delta}(x) = \lim_{n\to\infty} \widehat{G}_n(x)$, where

$$\widehat{G}_n(x) = \sum_{k=1}^{\infty} g_n(x-2\,k\,\pi) = \sum_{k=1}^{\infty} \frac{n}{\pi\big[1+n^2(x-2\,k\,\pi)^2\big]},$$

which can be proven to converge through application of the integral test.

(*b*) Let $h(x)$ be a $\mathrm{C}^\infty$ function with compact support: $\operatorname{supp} h \subset [a,b]$, so that $h(x) = 0$ for $x \le a$ or $x \ge b$. Then
$$\int_{-\infty}^{\infty} \widetilde{\delta}(x)\, u(x)\, dx = \sum_k u(2k\pi),$$
where the (finite) sum is over all multiples of $2\pi$ such that $a \le 2k\pi \le b$.

♢ 6.1.38. It suffices to note that if $u(x)$ is any smooth function on $[a,b]$, then
$$\lim_{n\to\infty} \int_a^b u(x) \cos n x \, dx = 0 \quad \text{by the Riemann–Lebesgue Lemma 3.40.} \qquad \textit{Q.E.D.}$$

6.1.40. (a) True.

---

6.2.1. To determine the Green's function, we must solve the boundary value problem
$$-c\,u'' = \delta(x-\xi), \qquad u(0) = 0, \qquad u'(1) = 0.$$
The general solution to the differential equation is
$$u(x) = -\frac{\rho(x-\xi)}{c} + a x + b, \qquad u'(x) = -\frac{\sigma(x-\xi)}{c} + a.$$
The integration constants $a, b$ are fixed by the boundary conditions
$$u(0) = b = 0, \qquad u'(1) = -\frac{1}{c} + a = 0.$$
Therefore, the Green's function for this problem is
$$G(x;\xi) = \begin{cases} x/c, & x \le \xi, \\ \xi/c, & x \ge \xi. \end{cases}$$
The superposition principle implies that the solution to the boundary value problem is
$$u(x) = \int_0^1 G(x;\xi) f(\xi)\, d\xi \;=\; \frac{1}{c}\int_0^x \xi\, f(\xi)\, d\xi \;+\; \frac{x}{c}\int_x^1 f(\xi)\, d\xi.$$
To verify the formula, we use formula (6.55) to compute
$$u'(x) = x\, f(x) - x\, f(x) + \frac{1}{c}\int_x^1 f(\xi)\, d\xi = \frac{1}{c}\int_x^1 f(\xi)\, d\xi, \qquad u''(x) = -\frac{1}{c} f(x).$$
Moreover,
$$u(0) = \frac{1}{c}\int_0^0 \xi\, f(\xi)\, d\xi + \frac{0}{c}\int_0^1 \xi\, f(\xi)\, d\xi = 0, \qquad u'(1) = \frac{1}{c}\int_1^1 f(\xi)\, d\xi = 0. \qquad \textit{Q.E.D.}$$

6.2.3. .5 mm — by linearity and symmetry of the Green's function.

6.2.9. True — the solution is $u(x) = 1$.

6.2.11. (a) $G(x;\xi) = \begin{cases} \dfrac{\sinh \omega x \, \cosh \omega (1-\xi)}{\omega \cosh \omega}, & x \le \xi, \\[2mm] \dfrac{\cosh \omega (1-x) \, \sinh \omega \xi}{\omega \cosh \omega}, & x \ge \xi. \end{cases}$ (*b*) If $x \le \frac{1}{2}$, then
$$\begin{aligned} u(x) &= \int_0^x \frac{\cosh \omega (1-x)\, \sinh \omega \xi}{\omega \cosh \omega}\, d\xi \;+\; \int_x^{1/2} \frac{\sinh \omega x\, \cosh \omega (1-\xi)}{\omega \cosh \omega}\, d\xi \\ &\qquad\qquad - \int_{1/2}^1 \frac{\sinh \omega x\, \cosh \omega (1-\xi)}{\omega \cosh \omega}\, d\xi \\ &= \frac{1}{\omega^2} - \frac{\left(e^{\omega/2} - e^{-\omega/2} + e^{-\omega}\right) e^{\omega x} + \left(e^{\omega} - e^{\omega/2} + e^{-\omega/2}\right) e^{-\omega x}}{\omega^2 (e^{\omega} + e^{-\omega})}, \end{aligned}$$

while if $x \geq \frac{1}{2}$, then

$$u(x) = \int_0^{1/2} \frac{\cosh\omega\,(1-x)\,\sinh\omega\,\xi}{\omega\cosh\omega}\,d\xi \;-\; \int_{1/2}^{x} \frac{\cosh\omega\,(1-x)\,\sinh\omega\,\xi}{\omega\cosh\omega}\,d\xi \;-\; \int_x^1 \frac{\sinh\omega\,x\,\cosh\omega\,(1-\xi)}{\omega\cosh\omega}\,d\xi$$

$$= -\frac{1}{\omega^2} + \frac{\left(e^{-\omega/2} - e^{-\omega} + e^{-3\omega/2}\right)e^{\omega x} + \left(e^{3\omega/2} - e^{\omega} + e^{\omega/2}\right)e^{-\omega x}}{\omega^2(e^{\omega} + e^{-\omega})}.$$

---

♢ 6.3.1. At a point $\mathbf{x} \in C_R$, the corresponding unit normal is $\mathbf{n} = \mathbf{x}/R$. Thus,

$$\frac{\partial f}{\partial \mathbf{n}} = \mathbf{n}\cdot\nabla f = \frac{\mathbf{x}}{R}\cdot\nabla f = \frac{x}{R}\frac{\partial f}{\partial x} + \frac{y}{R}\frac{\partial f}{\partial y} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}$$

by the chain rule. See also (4.104).

♢ 6.3.5. According to Exercise 6.1.13(*c*),

$$\delta(\beta\,x, \beta\,y) = \delta(\beta\,x)\,\delta(\beta\,y) = \frac{1}{|\,\beta\,|^2}\,\delta(x)\,\delta(y) = \frac{1}{\beta^2}\,\delta(x,y).$$

6.3.9. We rewrite $f(x,y) = \sigma(3x - 2y - 1)$ in terms of the step function. Thus, by the chain rule,

$$\frac{\partial f}{\partial x} = 3\,\delta(3x - 2y - 1) = \delta\left(x - \tfrac{2}{3}\,y - \tfrac{1}{3}\right), \qquad \frac{\partial f}{\partial y} = -2\,\delta(3x - 2y - 1) = -\,\delta\left(y - \tfrac{3}{2}\,y + \tfrac{1}{2}\right).$$

6.3.12. There is no equilibrium since (6.90) is not satisfied. Physically, you cannot remain in equilibrium while energy is continually flowing into the plate through its boundary.

---

6.3.15. (a)

$$u(r,\theta) = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{1/2} \log\left(\frac{1 + r^2\rho^2 - 2\,r\rho\cos(\theta-\varphi)}{r^2 + \rho^2 - 2\,r\rho\cos(\theta-\varphi)}\right)\rho\,d\rho\,d\varphi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{1/2} \log\left(\frac{1 + r^2\rho^2 - 2\,r\rho\cos\varphi}{r^2 + \rho^2 - 2\,r\rho\cos\varphi}\right)\rho\,d\rho\,d\varphi,$$

where the second expression follows upon replacing the integration variable $\varphi$ by $\varphi - \theta$. The latter formula does not depend on $\theta$, and hence the solution is radially symmetric, which is a consequence of the radial symmetry of the forcing function.

(*b*) Since the solution $u(r)$ is radially symmetric, it satisfies the ordinary differential equation

$$u_{rr} + \frac{1}{r}\,u_r = \begin{cases} 1, & r < \frac{1}{2}, \\ 0, & \frac{1}{2} < r < 1, \end{cases} \qquad u(1) = 0.$$

Setting $v = u_r$ reduces this to a first-order ordinary differential equation, with solution

$$u_r = v = \begin{cases} \frac{1}{2}\,r + a/r, & r < \frac{1}{2}, \\ b/r, & \frac{1}{2} < r < 1, \end{cases} = \begin{cases} \frac{1}{2}\,r, & r < \frac{1}{2}, \\ 1/(8r), & \frac{1}{2} < r < 1, \end{cases}$$

where the integration constant $a = 0$ because $u_r$ cannot have a singularity at the origin, while $b = \frac{1}{8}$ because $u_r$ is continuous at $r = \frac{1}{2}$. Integrating a second time produces the

solution:
$$u = \begin{cases} \frac{1}{4}r^2 + c, & r < \frac{1}{2}, \\ \frac{1}{8}\log r + d, & \frac{1}{2} < r < 1, \end{cases} \quad = \quad \begin{cases} \frac{1}{4}r^2 - \frac{1}{16} + \frac{1}{8}\log 2, & r < \frac{1}{2}, \\ \frac{1}{8}\log r, & \frac{1}{2} < r < 1, \end{cases}$$
where $d = 0$ due to the boundary condition $u(1) = 0$, while $c = \frac{1}{8}\log\frac{1}{2} - \frac{1}{16}$ because $u_r$ is continuous at $r = \frac{1}{2}$.

♡ 6.3.18. (a) Using the image point $(\xi, -\eta)$, we find $G(x,y;\xi,\eta) = \dfrac{1}{4\pi}\log\dfrac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}$.

(b) $u(x,y) = \dfrac{1}{4\pi}\displaystyle\int_0^\infty \int_{-\infty}^\infty \frac{1}{1+\eta}\log\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\,d\xi\,d\eta.$

♠ 6.3.25. (a) According to (6.116), the potential is
$$u(x,y) = -\frac{1}{4\pi}\int_{-1}^1 \int_{-1}^1 \log\left[\,(x-\xi)^2 + (y-\eta)^2\,\right] d\eta\,d\xi.$$
The gravitational force is its gradient $\nabla u$, with components
$$\frac{\partial u}{\partial x}(x,y) = -\frac{1}{2\pi}\int_{-1}^1 \int_{-1}^1 \frac{x-\xi}{(x-\xi)^2 + (y-\eta)^2}\,d\eta\,d\xi,$$
$$\frac{\partial u}{\partial x}(x,y) = -\frac{1}{2\pi}\int_{-1}^1 \int_{-1}^1 \frac{y-\eta}{(x-\xi)^2 + (y-\eta)^2}\,d\eta\,d\xi.$$

(b) Using numerical integration we find:
$$u(2,0) \approx -.4438, \qquad \nabla u(2,0) \approx (\,-.3134, 0\,)^T, \qquad \|\,\nabla u(2,0)\,\| \approx .3134,$$
$$u\big(\sqrt{2}\,,\sqrt{2}\,\big) \approx -.4385, \quad \nabla u\big(\sqrt{2}\,,\sqrt{2}\,\big) \approx (\,.2292, .2292\,)^T, \quad \|\,\nabla u\big(\sqrt{2}\,,\sqrt{2}\,\big)\,\| \approx .3241.$$
So the gravitational force at $\big(\sqrt{2}\,,\sqrt{2}\,\big)$ is slightly stronger, in part because it is closer to the edge of the square.

6.3.27. The solution
$$u(t,x) = \tfrac{1}{2}\,\delta(x - ct - a) + \tfrac{1}{2}\,\delta(x + ct - a) \qquad (*)$$
consists of two half-strength delta spikes traveling away from the starting position concentrated on the two characteristic lines. It is the limit of a sequence of classical solutions $u^{(n)}(t,x) \to u(t,x)$ as $n \to \infty$ which have initial conditions that converge to the delta function:
$$u^{(n)}(0,x) \;\longrightarrow\; \delta(x-a), \qquad u_t^{(n)}(0,x) = 0.$$
For example, using (6.10), the initial conditions
$$u^{(n)}(0,x) = \frac{n}{\pi(1 + n^2(x-a)^2)}$$
lead to the classical solutions
$$u^{(n)}(t,x) = \frac{n}{2\pi(1 + n^2(x-ct-a)^2)} \;+\; \frac{n}{2\pi(1 + n^2(x+ct-a)^2)}$$
that converge to the delta function solution $(*)$ as $n \to \infty$.

# Student Solutions to

## Chapter 7: Fourier Transforms

7.1.1. (*b*) $\sqrt{\dfrac{2}{\pi}}\,\dfrac{e^{\,\mathrm{i}\,k}}{k^2+1}\,;\qquad (d)\ \dfrac{\mathrm{i}}{\sqrt{2\pi}\,(k+3\,\mathrm{i}\,)} - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,(k-2\,\mathrm{i}\,)} = \dfrac{5}{\sqrt{2\pi}\,(k-2\,\mathrm{i}\,)(k+3\,\mathrm{i}\,)}\,.$

7.1.2. (*b*) $\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{x^2+1}\,.$

♢ 7.1.5. (a) $\sqrt{2\pi}\,\delta(k-\omega)$;

(*b*) $\mathcal{F}[\cos\omega\,x] = \sqrt{\dfrac{\pi}{2}}\,\bigl[\,\delta(k+\omega)+\delta(k-\omega)\,\bigr];\qquad \mathcal{F}[\sin\omega\,x] = \mathrm{i}\,\sqrt{\dfrac{\pi}{2}}\,\bigl[\,\delta(k+\omega)-\delta(k-\omega)\,\bigr].$

7.1.7.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a\cos k\,x + k\sin k\,x}{a^2+k^2}\,dk = \begin{cases} e^{-\,a\,x}, & x>0,\\ \frac{1}{2}\,, & x=0,\\ 0, & x<0,\end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a\sin k\,x - k\cos k\,x}{a^2+k^2}\,dk = 0.$$

The second identity follows from the fact that the integrand is odd.

♢ 7.1.12. (a) If $f(x)=f(-x)$, then, using Exercise 7.1.10(a), $\widehat{f}(k)=\widehat{f}(-k)$.

(*b*) If $f(x)=\overline{f(x)}$, then by Exercise 7.1.10(*b*), $\widehat{f}(k)=\overline{\widehat{f}(-k)}$; if, in addition, $f(x)$ is even, so is $\widehat{f}(k)$ and so $\widehat{f}(k)=\overline{\widehat{f}(k)}$ is real and even.

7.1.16. (a) $\widehat{f}(k-a)$.

♢ 7.1.17. (a) $f(x)\ \sim\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\widehat{f}_1(k)\,e^{\,\mathrm{i}\,k\,x}\,dk$ , (b) $\widehat{f}_1(k)=\sqrt{2\pi}\,\widehat{f}(k)$.

♡ 7.1.19. (*i*) Using Euler's formula (3.59)

$$\begin{aligned}\widehat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(\cos k\,x - \,\mathrm{i}\,\sin k\,x)\,dx\\ &= \int_{-\infty}^{\infty} f(x)\cos k\,x\,dx - \,\mathrm{i}\int_{-\infty}^{\infty} f(x)\sin k\,x\,dx = \widehat{c}\,(k) - \,\mathrm{i}\,\widehat{s}\,(k).\end{aligned}$$

(*ii*) (*b*) $\widehat{c}\,(k)=\sqrt{\dfrac{2}{\pi}}\,\dfrac{\cos k}{k^2+1}\,,\qquad \widehat{s}\,(k) = -\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin k}{k^2+1}\,.$

♢ 7.1.20. (a) (*i*) $\dfrac{2}{\pi(k^2+1)(l^2+1)}\,.\qquad (c)\ f(x,y) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(k,l)\,e^{\,\mathrm{i}\,(k\,x+l\,y)}\,dl\,dk.$

---

7.2.1. (a) $e^{-\,k^2/2}$, (*d*) $\mathrm{i}\,\sqrt{2\pi}\,\delta'(k)$.

7.2.3. (b) $\dfrac{\mathrm{i}}{2\sqrt{2}}\, x\, e^{-x^2/4}$.

♢ 7.2.8. $\displaystyle -\mathrm{i}\left(\int_{-\infty}^{k} \widehat{f}(l)\, dl - \frac{1}{2}\int_{-\infty}^{\infty} \widehat{f}(k)\, dk\right) = -\mathrm{i}\left(\int_{-\infty}^{k} \widehat{f}(l)\, dl - \sqrt{\frac{\pi}{2}}\, f(0)\right).$

7.2.9. (a) $-\mathrm{i}\sqrt{\dfrac{\pi}{2}}\ \operatorname{sign} k$.

7.2.13. (a) $\sqrt{\dfrac{\pi}{2}}\,\delta(k+1) - \sqrt{2\pi}\,\delta(k) + \sqrt{\dfrac{\pi}{2}}\,\delta(k-1)$.

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7.3.1. (a) $\dfrac{\mathrm{i}}{7}\sqrt{\dfrac{\pi}{2}}\,\left(e^{6\,\mathrm{i}\,x} - e^{-\,\mathrm{i}\,x}\right)\operatorname{sign} x$.

7.3.3. (b) $\mathrm{i}\sqrt{\dfrac{\pi}{2}}\,(e^{-\,|\,k\,|} - 1)\operatorname{sign} k$.

♢ 7.3.6. (a) $u(x) = \frac{1}{2}\, e^{-\,|\,x\,|}\big(1 + |\,x\,|\big)$.

(b) Using l'Hôpital's Rule:

$$\lim_{\omega\to 1} \frac{e^{-\,|\,x\,|} - \omega^{-1}\, e^{-\omega\,|\,x\,|}}{\omega^2 - 1} = \lim_{\omega\to 1} \frac{|\,x\,|\, e^{-\omega\,|\,x\,|} + \omega^{-2}\, e^{-\omega\,|\,x\,|}}{2\,\omega} = \tfrac{1}{2}\,(1 + |\,x\,|\,)\, e^{-\,|\,x\,|}.$$

---

7.3.10. (a) $\widehat{h}(k) = \dfrac{\sqrt{2}}{k^2+1}\, e^{-\,k^2/4}$;

(b) $h(x) = \dfrac{\sqrt{\pi}\,\sqrt[4]{e}}{2}\left(e^{-\,x}\Big[1 - \operatorname{erf}\big(\tfrac{1}{2} - x\big)\Big] + e^{x}\Big[1 - \operatorname{erf}\big(\tfrac{1}{2} + x\big)\Big]\right).$

7.3.14. (a) $\widehat{f}(k) = \dfrac{1 + e^{-\,\mathrm{i}\,\pi k}}{\sqrt{2\pi}\,(1-k^2)}, \qquad \widehat{g}(k) = \dfrac{\mathrm{i}\,k(1 + e^{-\,\mathrm{i}\,\pi k})}{\sqrt{2\pi}\,(1-k^2)}$;

(b) $h(x) = \left(\tfrac{1}{2}\,\pi - \tfrac{1}{2}\,|\,x - \pi\,|\right)\sin x$; (c) $\widehat{h}(k) = \sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k) = \dfrac{\mathrm{i}\,k(1 + e^{-\,\mathrm{i}\,\pi k})^2}{\sqrt{2\pi}\,(1-k^2)^2}$.

♢ 7.3.22. (b)
$$\begin{aligned} f * [\,a\, g + b\, h\,](x) &= \int_{-\infty}^{\infty} f(x-\xi)\,[\,a\, g(\xi) + b\, h(\xi)\,]\, d\xi \\ &= a\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi + b\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = a[\,f * g(x)\,] + b[\,f * h(x)\,]. \end{aligned}$$

The second bilinearity identity is proved by a similar computation, or by using the symmetry property:

$$(a\, f + b\, g) * h = h * (a\, f + b\, g) = a\,(h * f) + b\,(h * g) = a\,(f * h) + b\,(g * h).$$

(d) $\displaystyle f * 0 = \int_{-\infty}^{\infty} f(x-\xi)\, 0\, d\xi = 0.$

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7.4.1. (a) $2 = \displaystyle\int_{-1}^{1} dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 k}{k^2}\, dk;$

(b) Since the integrand is even, $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}\,.$

♢ 7.4.6.

$$\|\, f \,\|^2 = \int_{-\infty}^{\infty} |\, f(x)\,|^2\, dx = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{n-n^{-2}}^{n+n^{-2}} dx = 2\sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{2}{3}\,\pi^2 < \infty,$$

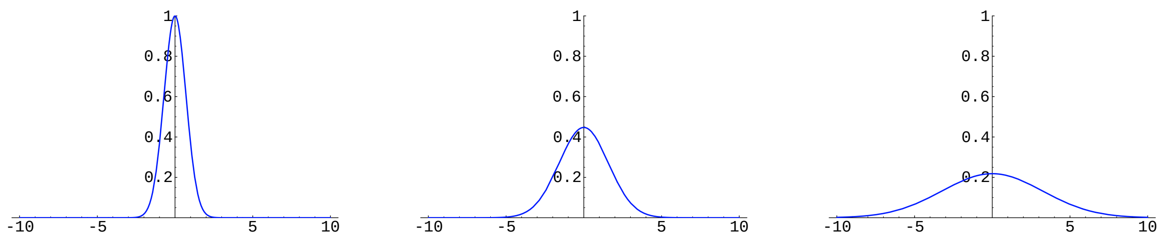
and so $f \in \mathrm{L}^2$. However $f(x) \not\to 0$ as $x \to \pm\infty$ since $f(n) = 1$ for arbitrarily large (positive and negative) integers $n$.

7.4.9. (a) $a = 1$; (b) $\|\, x\,\varphi(x)\,\|\; \|\,\varphi'(x)\,\| = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \geq \frac{1}{2}\,.$

# Student Solutions to

## Chapter 8: Linear and Nonlinear Evolution Equations

8.1.1. (a) $u(t,x) = \dfrac{1}{\sqrt{4t+1}}\, e^{-x^2/(4t+1)}$;



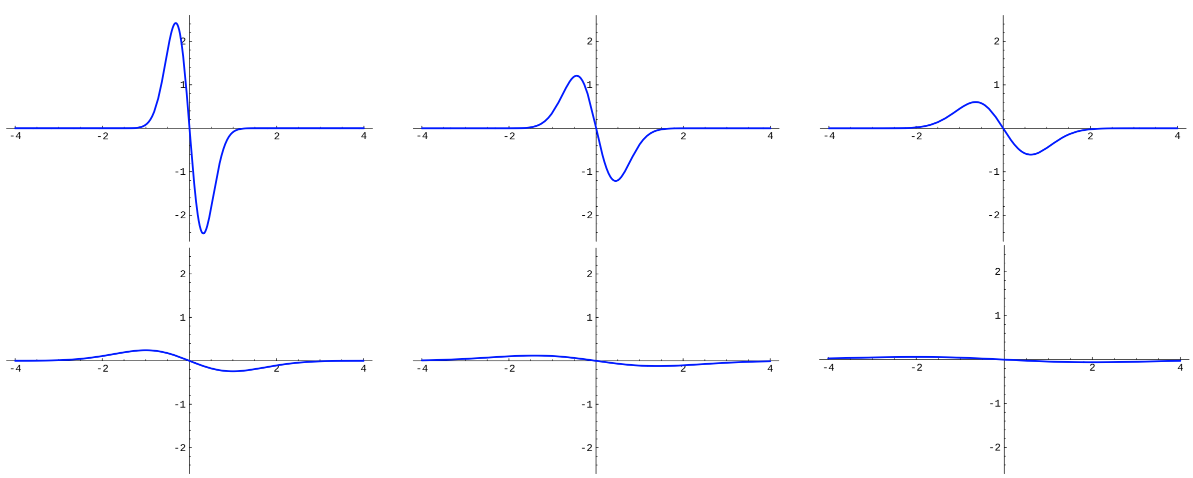

8.1.6. (a) The maximum occurs at $x = \xi$, where $F(t,\xi;\xi) = \dfrac{1}{2\sqrt{\pi\, t}}$.

(b) One justification is to look at where the solution has half its maximal value, which occurs at $x = \xi \pm 2\sqrt{t \log 2}$, and so, under this measure, the width is $4\sqrt{t \log 2}$. Alternatively, the width can be measured by the standard deviation. In general, the Gaussian distribution $\dfrac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\xi)^2/(2\sigma^2)}$ has mean $\xi$ and standard deviation $\sigma$. Comparing with the fundamental solution (8.14), we find $\sigma = \sqrt{2t}$.

8.1.10. (a) For the $x$ derivative:

$$\frac{\partial F}{\partial x}(t,x;\xi) = \frac{\xi - x}{4\sqrt{\pi}\, t^{3/2}}\, e^{-(x-\xi)^2/(4t)} \quad \text{has initial condition } u(0,x) = \delta'(x-\xi).$$

(b) Plots of $\dfrac{\partial F}{\partial x}(t,x;0)$ at times $t = .05, .1, .2, .5, 1., 2.$:




(c) $$\frac{\partial F}{\partial x}(t,x;\xi) = \frac{\xi - x}{4\sqrt{\pi}\, t^{3/2}}\, e^{-(x-\xi)^2/(4t)} = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} k\, e^{-k^2 t}\, e^{\mathrm{i}\,k(x-\xi)}\, dk.$$

8.1.17. $F(t,x;\xi) = \dfrac{1}{2\sqrt{\pi\gamma t}}\, e^{-(x-\xi)^2/(4\gamma t)-\alpha t}$.

8.1.22. (a) $\|\, e^{-a k^2}\, \widehat{f}(k)\,\|^2 = \displaystyle\int_{-\infty}^{\infty} e^{-2ak^2}\, |\,\widehat{f}(k)\,|^2\, dk \le \int_{-\infty}^{\infty} |\,\widehat{f}(k)\,|^2\, dk = \|\,\widehat{f}(k)\,\|^2 < \infty$.

(*b*) This follows immediately from part (a) and the Plancherel formula (7.64).

8.1.23. In this case, the initial condition is

$$z(0,y) = e^{(\kappa-1)y/2} \max\{\, p - e^y, 0\,\},$$

and so

$$\begin{aligned} z(\tau,y) &= \frac{1}{2\sqrt{\pi\tau}} \int_0^{\log p} e^{-(y-\eta)^2/(4\tau)+(\kappa-1)\eta/2}(p-e^\eta)\, d\eta \\ &= \frac{1}{2}\left[ e^{(\kappa+1)^2\tau/4+(\kappa+1)y/2}\, \operatorname{erfc}\left( \frac{(\kappa+1)\tau + y - \log p}{2\sqrt{\tau}} \right) \right. \\ &\qquad \left. - \, p\, e^{(\kappa-1)^2\tau/4+(\kappa-1)y/2}\, \operatorname{erfc}\left( \frac{(\kappa-1)\tau + y - \log p}{2\sqrt{\tau}} \right) \right]. \end{aligned}$$

Thus,

$$\begin{aligned} u(t,x) = \frac{1}{2}\left[ p\, e^{-r(t_\star - t)}\, \operatorname{erfc}\left( \frac{\left(r - \frac{1}{2}\sigma^2\right)(t_\star - t) + \log(x/p)}{\sqrt{2\sigma^2(t_\star - t)}} \right) \right. \\ \left. - \, x\, \operatorname{erfc}\left( \frac{\left(r + \frac{1}{2}\sigma^2\right)(t_\star - t) + \log(x/p)}{\sqrt{2\sigma^2(t_\star - t)}} \right) \right]. \end{aligned}$$

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8.2.1. 92 minutes.

8.2.5. $U(t,x) = \frac{5}{9}[\,u(t,x) - 32\,] + 273.15 = \frac{5}{9}\,u(t,x) + 255.372$. Changing the temperature scale does not alter the diffusion coefficient.

8.2.8. Using time translation symmetry, $u(t,x) = u^\star(t+1,x)$ also solves the heat equation and satisfies $u(0,x) = u^\star(1,x) = f(x)$.

8.2.9. (a) The fundamental solution $F(t,x) = \dfrac{1}{2\sqrt{\pi t}}\, e^{-x^2/(4t)}$ satisfies $F(1,x) = \dfrac{1}{2\sqrt{\pi}}\, e^{-x^2/4}$. Therefore, by Exercise 8.2.8,

$$u(t,x) = 2\sqrt{\pi}\, F(t+1,x) = \frac{1}{\sqrt{t+1}}\, e^{-x^2/[\,4(t+1)\,]}.$$

8.2.11. (*b*) Scaling symmetries: $U(t,x) = \beta^{(c-1)/2}\, u(\beta^{-1} t, \beta^{-c} x)$ for any constant $c$; similarity ansatz: $u(t,x) = t^{(c-1)/2} v(\xi)$ where $\xi = x\, t^{-c}$; reduced ordinary differential equation: $(v^2 - c\xi)\, v' + \frac{1}{2}(c-1)\, v = 0$. If $c = 1$, then $v = \sqrt{\xi}$ or constant, and $u(t,x) = \sqrt{x/t}$ or constant. For $c \ne 1$, the implicit solution is $\xi = v^2 + k\, v^{2+2/(c-1)}$ where $k$ is the integration constant, and so $x = t\, u^2 + k\, u^{2c/(c-1)}$.

8.2.12. (a) Set $U(t,x) = u(t-a,x)$. Then, by the chain rule,

$$\frac{\partial^2 U}{\partial t^2} - c^2\, \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - c^2\, \frac{\partial^2 u}{\partial x^2} = 0.$$

8.2.15. (a) Scaling symmetries: $(x,y,u) \longmapsto (\beta\,x,\beta\,y,\beta^c u)$ for any constant $c$, producing the rescaled solution $U(t,x) = \beta^c\,u(\beta^{-1}\,x,\beta^{-1}\,y)$.

(b) The similarity ansatz is $u(t,x) = x^c\,v(\xi)$ where $\xi = y/x$. Substituting into the Laplace equation produces the reduced ordinary differential equation

$$(\xi^2+1)v'' - 2(c-1)\xi\,v' + c(c-1)v = 0.$$

(c) The general solution to the reduced ordinary differential equation can be written as $v(\xi) = \operatorname{Re}[\,k\,(1+\mathrm{i}\,\xi)^c\,]$, where $k = k_1 + \mathrm{i}\,k_2$ is an arbitrary complex constant. The corresponding similarity solutions to the Laplace equation are $u(x,y) = \operatorname{Re}[\,k\,(x+\mathrm{i}\,y)^c\,]$. In particular, if $c = n$ is an integer, one recovers the harmonic polynomials of degree $n$.

---

8.3.1. True. This follows immediately from Corollary 8.7, with $m > 0$ being the minimum of the initial and boundary temperatures.

8.3.3. First note that $M(t) \geq 0$ for $t > 0$, since $u(t,a) = u(t,b) = 0$. Given $0 < t_1 < t_2$, the Maximum Principle applied to the rectangle $R = \{\,t_1 \leq t \leq t_2,\ a \leq x \leq b\,\}$ implies that the maximum of $u(t,x)$ on $R$ equals $M(t_1) \geq 0$. Therefore,

$$M(t_2) = \max\{\,u(t_2,x)\mid\ a \leq x \leq b\,\} \leq M(t_1).$$

---

8.4.1. (b)

$$u(t,x) = 1 - \frac{1+\operatorname{erf}\left(\dfrac{x}{2\sqrt{t}}\right)}{1+\operatorname{erf}\left(\dfrac{x}{2\sqrt{t}}\right) + e^{t/4-x/2}\left[1-\operatorname{erf}\left(\dfrac{x-t}{2\sqrt{t}}\right)\right]}$$

$$= \frac{1-\operatorname{erf}\left(\dfrac{x-t}{2\sqrt{t}}\right)}{1-\operatorname{erf}\left(\dfrac{x-t}{2\sqrt{t}}\right) + e^{x/2-t/4}\left[1+\operatorname{erf}\left(\dfrac{x}{2\sqrt{t}}\right)\right]},$$

8.4.8. (a) Setting $U(t,x) = \lambda\,u(\alpha^{-1}\,t,\beta^{-1}\,x)$, we find

$$\frac{\partial U}{\partial t} = \frac{\lambda}{\alpha}\,\frac{\partial u}{\partial t}, \qquad \frac{\partial U}{\partial x} = \frac{\lambda}{\beta}\,\frac{\partial u}{\partial x}, \qquad \frac{\partial^2 U}{\partial x^2} = \frac{\lambda}{\beta^2}\,\frac{\partial^2 u}{\partial x^2}.$$

Thus $U(t,x)$ solves the rescaled Burgers' equation

$$U_t + \frac{\beta}{\alpha\,\lambda}\,U\,U_x = \frac{\beta^2\gamma}{\alpha}\,U_{xx}.$$

(b) In light of part (a), setting $\alpha = 1$, $\beta = \sqrt{\dfrac{\sigma}{\gamma}}$, $\lambda = \dfrac{1}{\rho}\sqrt{\dfrac{\sigma}{\gamma}}$, we find that

$$U(t,x) = \lambda\,u(t,\beta^{-1}\,x) = \frac{1}{\rho}\sqrt{\frac{\sigma}{\gamma}}\;u\left(t,\sqrt{\frac{\sigma}{\gamma}}\,x\right),$$

where $u(t,x)$ solves the initial value problem

$$u_t + u\,u_x = \gamma\,u_{xx}, \qquad u(0,x) = f(x) = \frac{1}{\rho}\sqrt{\frac{\sigma}{\gamma}}\;F\left(\sqrt{\frac{\sigma}{\gamma}}\,x\right).$$

Thus, $u(t,x)$ is given by (8.84), from which one can can reconstruct the solution $U(t,x)$ by the preceding formula.

8.4.10. (*b*) $\varphi_t = \gamma\,[\,\varphi_{xx} - \varphi_x^2/(2\,\varphi)\,]$.

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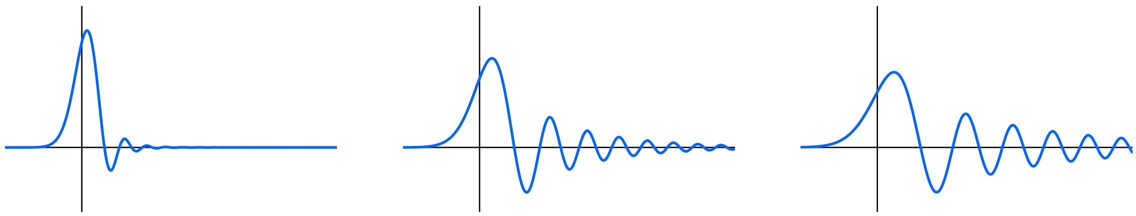
8.5.1. Since $u(-t,-x)$ solves the dispersive equation, the solution is a mirror image of its values in positive time. Thus, the solution profiles are



$t = -.1$ $\qquad$ $t = -.5$ $\qquad$ $t = -1$

8.5.4. (*b*) Dispersion relation: $\omega = -\,k^5$; phase velocity: $c_p = -\,k^4$;
group velocity: $c_g = -5\,k^4$; dispersive.

♢ 8.5.7. (a) Conservation of mass:

$$\frac{\partial}{\partial t}\,u + \frac{\partial}{\partial x}\,u_{xx} = 0,$$

and hence the mass flux is $X = u_{xx}$. We conclude that the total mass

$$\int_{-\infty}^{\infty} u(t,x)\,dx = \text{constant},$$

provided the flux goes to zero at large distances: $u_{xx} \to 0$ as $|\,x\,| \to \infty$.

---

8.5.12. (a) Using the chain rule, $U_t = u_t - c\,u_x = -\,u_{xxx} - (u+c)\,u_x = -\,U_{xxx} - U\,U_x$.

(*b*) It can be interpreted as the effect of a Galilean boost to a moving coordinate frame with velocity $c$; the only change in the solution is to shift its height above the $x$ axis.

8.5.16. (a) $u(t,x) = \sqrt{6\,c}\ \mathrm{sech}\,[\,\sqrt{c}\,(x-c\,t) + \delta\,]$; (*b*) the amplitude is proportional to the square root of the speed, while the width is proportional to $1/\sqrt{c}$.

♢ 8.5.18. (a) The corresponding flux is $X_1 = u_{xx} + \frac{1}{2}\,u^2$, and the conservation law is

$$\frac{\partial T_1}{\partial t} + \frac{\partial X_1}{\partial x} = u_t + u_{xxx} + u\,u_x = 0.$$

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# Student Solutions to

## Chapter 9: A General Framework for Linear Partial Differential Equations

9.1.1. (a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{13}{7} & -\frac{10}{7} \\ \frac{5}{7} & \frac{15}{7} \end{pmatrix}$.

9.1.2. Domain (a), target (b): $\begin{pmatrix} 2 & -3 \\ 4 & 9 \end{pmatrix}$; domain (b), target (c): $\begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ \frac{1}{3} & \frac{10}{3} \end{pmatrix}$.

9.1.3. (b) $\begin{pmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & \frac{2}{3} & 2 \end{pmatrix}$.

9.1.4. Domain (a), target (b): $\begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 0 & 2 & 6 \end{pmatrix}$.

9.1.5. Domain (a), target (a): $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$; domain (b), target (c): $\begin{pmatrix} 1 & 0 & -1 \\ \frac{8}{3} & \frac{8}{3} & \frac{4}{3} \end{pmatrix}$.

9.1.9. (a) $L^*[v] = -\dfrac{d}{dx}\big[\,x\,v(x)\big] + v(x) = -x\,v'(x)$.

♢ 9.1.13. (a) Given $L\colon U \to V$, for any $u \in U$, $v_1, v_2 \in V$, $c_1, c_2 \in \mathbb{R}$, we use (9.2) to compute

$$\begin{aligned} \langle u\,; L^*[c_1 v_1 + c_2 v_2]\,\rangle &= \langle\!\langle\, L[u]\,; c_1 v_1 + c_2 v_2 \,\rangle\!\rangle = c_1 \langle\!\langle\, L[u]\,; v_1 \,\rangle\!\rangle + c_2 \langle\!\langle\, L[u]\,; v_2 \,\rangle\!\rangle \\ &= c_1 \langle\, u\,; L^*[v_1]\,\rangle + c_2 \langle\, u\,; L^*[v_2]\,\rangle = \langle\, u\,; c_1 L^*[v_1] + c_2 L^*[v_2]\,\rangle. \end{aligned}$$

Since this holds for all $u \in U$, we conclude that

$$L^*[c_1 v_1 + c_2 v_2] = c_1 L^*[v_1] + c_2 L^*[v_2].$$

♢ 9.1.15. Given $u \in U$ and $v \in V$, we have

$$\langle\!\langle\, (L^*)^*[u]\,; v \,\rangle\!\rangle = \langle\, u\,; L^*[v]\,\rangle = \langle\!\langle\, L[u]\,; v \,\rangle\!\rangle.$$

Since this holds for all $u$ and $v$, we conclude that $(L^*)^* = L$. *Q.E.D.*

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9.1.18. (b) The cokernel of $A = \begin{pmatrix} 6 & -3 & 9 \\ 2 & -1 & 3 \end{pmatrix}$ has basis $\mathbf{v} = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$. Since $\mathbf{v} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 0$, the system is compatible. The general solution is $x = \frac{1}{2}\,y - \frac{3}{2}\,z + 1$, where $y, z$ are arbitrary.

9.1.19. (a) $2\,a - b + c = 0$.

9.1.21. Under the $\mathrm{L}^2$ inner product, the adjoint system $x\,v'' + v' = 0$, $v'(1) = v'(2) = 0$, has constant solutions, so the Fredholm constraint is $\langle\, 1\,; 1 - \frac{2}{3}\,x\,\rangle = \int_1^2 \big(1 - \frac{2}{3}\,x\big)\,dx = 0$. Writing the equation as $D(x\,u') = 1 - \frac{2}{3}\,x$, we have $u' = 1 - \frac{1}{3}\,x + c/x$, with the boundary conditions requiring $c = -\frac{2}{3}$. Thus, the solution is $u(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\log x + a$, where $a$ is an arbitrary constant.

9.1.25. Since $(\nabla\cdot)^* = -\nabla$, the homogeneous adjoint problem is $-\nabla u = \mathbf{0}$ in $\Omega$ with no boundary conditions. Since $\Omega$ is connected, every solutions is constant, and therefore, the Fredholm Alternative requires $0 = \langle f\,;1\rangle = \iint_\Omega f(x,y)\,dx\,dy$.

---

9.2.1. (a) Self-adjoint; (c) not self-adjoint.

9.2.2. (i) (a) Self-adjoint; (c) not self-adjoint.

♢ 9.2.5. (a) Since $C^T = C$, we have $J^T = K^T C = J = C\,K$ if and only if $K$ satisfies the requirement of Example 9.15.

(b) By definition, $K > 0$ if and only if

$$0 < \langle\,\mathbf{u}\,;K\,\mathbf{u}\,\rangle = \mathbf{u}^T C\,K\,\mathbf{u} = \mathbf{u}^T J\,\mathbf{u} \qquad \text{for all} \qquad \mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n,$$

which holds if and only if $J > 0$ with respect to the dot product. *Q.E.D.*

9.2.10. We need to impose two boundary conditions at each endpoint. Some common possibilities are to require either

$$u(a) = v(a) = 0 \quad\text{or}\quad u(a) = v'(a) = 0 \quad\text{or}\quad v(a) = v'(a) = 0 \quad\text{or}\quad u'(a) = v'(a) = 0$$

at the left hand endpoint, along with a second pair

$$u(b) = v(b) = 0 \quad\text{or}\quad u(b) = v'(b) = 0 \quad\text{or}\quad v(b) = v'(b) = 0 \quad\text{or}\quad u'(b) = v'(b) = 0$$

at the other end. One can mix or match the options in any combination. Once we identify $v(x) = u''(x)$, this produces 4 boundary conditions on the functions $u$ in the domain of $S$, which always positive semi-definite, and is positive definite if and only if at least one of the boundary conditions requires that $u$ vanish at one of the endpoints.

♡ 9.2.13. (a) We define $\nabla\colon V \to W$, where $V$ is the vector space consisting of scalar functions $u(x,y)$ defined for $0 < x < a$, $0 < y < b$, and satisfying

$$u(x,0) = u(x,b), \qquad u_y(x,0) = u_y(x,b), \qquad u(0,y) = u(a,y), \qquad u_x(0,y) = u_x(a,y),$$

while $W$ is the set of vector fields

$$\mathbf{v}(x,y) = (\,v_1(x,y), v_2(x,y)\,)^T \quad \text{satisfying} \quad v_2(x,0) = v_2(x,b), \qquad v_1(0,y) = v_2(a,y).$$

The boundary integral in the basic integration by parts identity (9.33) reduces to

$$\oint_{\partial\Omega}\big(-u\,v_2\,dx + u\,v_1\,dy\big) = -\int_0^a u(x,0)\,v_2(x,0)\,dx + \int_0^b u(a,y)\,v_2(a,y)\,dy$$
$$+\int_0^a u(x,b)\,v_2(x,b)\,dx - \int_0^b u(0,y)\,v_2(0,y)\,dy = 0,$$

by the boundary conditions. This is the key point to proving (9.27), and hence writing the boundary value problem in self-adjoint form (9.60).

(b) The problem is *not* positive definite, because any constant function satisfies the boundary conditions, and hence belongs to $\ker\nabla \neq \{0\}$.

(c) By the Fredholm Alternative, $f$ must be orthogonal to the kernel, and hence must satisfy the condition

$$\langle\,f\,;1\,\rangle = \int_0^a\int_0^b f(x,y)\,dy\,dx = 0.$$

9.3.1. (a) $u_\star(x) = \frac{1}{6}\,x - \frac{1}{6}\,x^3$; (b) $Q[u] = \int_0^1 \left[\, \frac{1}{2}\,(u')^2 - x\,u \,\right] dx$, $u(0) = u(1) = 0$;

(c) $Q[u_\star] = -\frac{1}{90} = -.01111$; (d) $Q[cx - cx^3] = \frac{2}{5}\,c^2 - \frac{2}{15}\,c > -\frac{1}{90}$ for constant $c \neq \frac{1}{6}$, while, for example, $Q[cx - cx^2] = \frac{1}{6}\,c^2 - \frac{1}{12}\,c \geq -\frac{1}{96} = -.01042 > -\frac{1}{90}$, for all $c$, and $Q[c\sin\pi x] = \dfrac{\pi^2 c^2}{4} - \dfrac{c}{\pi} \geq -\dfrac{1}{\pi^4} = -.01027 > -\frac{1}{90}$, also for all $c$.

9.3.4. (b) Boundary value problem: $-((x+1)\,u')' = 5$, $u(0) = u(1) = 0$;
solution/minimizer: $u_\star(x) = \dfrac{5}{\log 2}\log(x+1) - 5\,x$.

9.3.5. (a) Unique minimizer: $u_\star(x) = \dfrac{1}{2}\,x^2 - 2\,x + \dfrac{3}{2} + \dfrac{\log x}{2\log 2}$.

(d) No minimizer since $1 - x^2$ is not positive for all $-2 < x < 2$.

9.3.7. $u(x) = \dfrac{1}{9} - \dfrac{e^{3x/2} + e^{3-3x/2}}{9(e^3+1)}$; the solution is unique.

9.3.9. (b) (i) $-\dfrac{d}{dx}\left( x\,\dfrac{du}{dx} \right) + 2\,u = 1$.

(ii) Minimize $Q[u] = \int_1^2 \left[\, \frac{1}{2}\,x\,u'(x)^2 + \frac{1}{2}\,u(x)^2 - u(x) \,\right] dx$ with $u(1) = u(2) = 0$.

9.3.15. $u(x) = x^2$ satisfies $\int_0^1 u''(x)\,u(x)\,dx = \frac{2}{3}$. Positivity of $\int_0^1 \left[ -u''(x)\,u(x) \right] dx$ holds only for functions that satisfy the boundary conditions $u(0) = u(1)$.

---

♡ 9.3.18. (a) First, by direct calculation,

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x^2 + y^2 - x - y.$$

Moreover, $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$.

(b) $Q[u] = \int_0^1 \int_0^1 \left[\, \frac{1}{2}\,\|\,\nabla u\,\|^2 - (x^2 + y^2 - x - y)\,u \,\right] dx\,dy = -\frac{1}{360} \approx -.002778$.

(c) For example, $Q[x\,y\,(1-x)(1-y)] = 0$, $Q[x^2 y\,(1-x)(1-y)] = -\frac{11}{6300} \approx -.001746$, $Q\left[ -\dfrac{32}{\pi^6}\sin\pi x\sin\pi y \right] = -\dfrac{256}{\pi^{10}} \approx -0.002734$.

9.3.23. Solving the corresponding boundary value problem

$$-\frac{d}{dx}\left( x\,\frac{du}{dx} \right) = -x^2, \quad u(1) = 0, \quad u(2) = 1, \qquad \text{yields} \qquad u(x) = \frac{x^3 - 1}{9} + \frac{2\,\log x}{9\,\log 2}.$$

---

9.4.1. (b) Eigenvalues: $7, 3$; eigenvectors: $\dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

9.4.2. (a) Eigenvalues $\frac{5}{2} \pm \frac{1}{2}\sqrt{17}$; positive definite.

9.4.5. (a) The minimum value is the smallest eigenvalue corresponding to the boundary value problem $-v'' = \lambda v$ subject to the indicated boundary conditions: minimum $= \pi^2$, eigenfunction $v(x) = \sin \pi x$.

♡ 9.4.9. (a) Eigenfunctions: $u_n(x) = \sin(n\pi \log x)$; eigenvalues: $\lambda_n = n^2\pi^2$.

(b) First note that the differential equation is in weighted Sturm–Liouville form (9.78) with $p(x) = x$, $\rho(x) = 1/x$, $q(x) = 0$. Therefore, the relevant inner product is

$$\langle f \,; g \rangle = \int_1^e \frac{f(x)\, g(x)}{x}\, dx.$$

Indeed, the change of variables $y = \log x$ shows that

$$\langle u_m \,; u_m \rangle = \int_1^e \frac{\sin(m\pi \log x)\ \sin(n\pi \log x)}{x}\, dx = 0 \qquad \text{for} \qquad m \neq n.$$

(c) $f(x) \sim \displaystyle\sum_{n=1}^{\infty} c_n \sin(n\pi \log x)$, where $c_n = \dfrac{\langle f \,; u_n \rangle}{\| u_n \|^2} = 2 \displaystyle\int_0^1 \frac{f(x) \sin(n\pi \log x)}{x}\, dx.$

(d) Closed form:

$$G(x;\xi) = \begin{cases} \xi^{-1}(1 - \log \xi) \log x, & 1 \le x \le \xi, \\ \xi^{-1}(1 - \log x) \log \xi, & \xi \le x \le e. \end{cases}$$

Eigenfunction series:

$$G(x;\xi) = \sum_{n=1}^{\infty} \frac{2 \sin(n\pi \log x) \sin(n\pi \log \xi)}{n^2 \pi^2 \xi}\,.$$

(e) The Green's function is not symmetric, but the modified Green's function is:

$$\widehat{G}(x;\xi) = \widehat{G}(\xi;x) = \xi\, G(x;\xi) = \begin{cases} (1 - \log \xi) \log x, & 1 \le x \le \xi, \\ (1 - \log x) \log \xi, & \xi \le x \le e. \end{cases}$$

(f) The double norm of the modified Green's function is

$$\| \widehat{G} \|^2 = \int_1^e \!\! \int_1^e \frac{\widehat{G}(x;\xi)^2}{x\,\xi}\, dx\, d\xi = 2 \int_1^e \!\! \int_1^\xi \frac{(1 - \log \xi)^2 (\log x)^2}{x\,\xi}\, dx\, d\xi = \frac{1}{90} < \infty.$$

Theorem 9.47 implies completeness of the eigenfunctions.

9.4.16. (a)

$$G(x,y;\xi,\eta) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin n\pi y\ \sin n\pi \eta}{m^2} + \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos m\pi x\ \sin n\pi y\ \cos m\pi \xi\ \sin n\pi \eta}{m^2 + n^2}\,.$$

---

9.5.1. The eigenfunctions are $v_k(x) = \sin k\pi x$ with eigenvalues $\lambda_k = \gamma \pi^2 k^2$ and norms $\| v_k \|^2 = \displaystyle\int_0^1 \sin^2 k\pi x\, dx = \tfrac{1}{2}$, for $k = 1, 2, \ldots$ . Therefore, by (9.128),

$$F(t,x;\xi) = \sum_{k=1}^{\infty} 2\, e^{-\gamma \pi^2 k^2 t} \sin k\pi x\ \sin k\pi \xi.$$

9.5.4. The eigenfunction boundary value problem is

$$v'''' = \lambda\, v, \qquad v(0) = v''(0) = v(1) = v''(1) = 0.$$

The eigenfunctions are $v_k(x) = \sin k\pi x$ with eigenvalues $\lambda_k = k^4 \pi^4$ for $k = 1, 2, \ldots$ . The solution to the initial value problem is

$$u(t,x) = \sum_{k=1}^{\infty} b_k\, e^{-k^4 \pi^4 t} \sin k\pi x,$$

where $b_k = 2\int_0^1 f(x)\sin k\pi x\,dx$ are the Fourier sine coefficients of $f(x)$ on $[0,1]$. The equilibrium state is $u(t,x) \to 0$, and the decay is exponentially fast, at a rate $\pi^4$, as given by the smallest eigenvalue.

---

9.5.9. (*b*) resonant, since $\langle\, 1\,;\sin 3\pi x\,\rangle = \displaystyle\int_0^1 \sin 3\pi x\,dx = \frac{2}{3\pi} \neq 0$;

(*d*) resonant, since $\langle\, \sin\pi x\,;\sin\pi x\,\rangle = \frac{1}{2} \neq 0$.

♢ 9.5.14. For nonresonant $\omega \neq \omega_k = k\pi c$,

$$u(t,x) = \frac{\cos\omega t\ \sin k\pi x - \cos k\pi c t\ \sin k\pi x}{\omega^2 - k^2\pi^2 c^2} + \sum_{k=1}^{\infty}\Big[\, b_k \cos k\pi c t\ \sin k\pi x + d_k \sin k\pi c t\ \sin k\pi x\,\Big],$$

whereas for resonant $\omega = \omega_k = k\pi c$,

$$u(t,x) = \frac{t\,\sin k\pi c t\ \sin k\pi x}{2k\pi c} + \sum_{k=1}^{\infty}\Big[\, b_k \cos k\pi c t\ \sin k\pi x + d_k \sin k\pi c t\ \sin k\pi x\,\Big],$$

where, in both cases,

$$b_k = 2\int_0^1 f(x)\,\sin k\pi x\,dx, \qquad d_k = \frac{2}{k\pi c}\int_0^1 g(x)\,\sin k\pi x\,dx,$$

are the Fourier sine coefficients of the initial displacement and velocity.

♢ 9.5.16. The function

$$\widetilde{u}(t,x) = u(t,x) - \left(\alpha(t) + \frac{\beta(t)-\alpha(t)}{\ell}\,x\right),$$

satisfies the initial-boundary value problem

$$\widetilde{u}_{tt} = c^2\widetilde{u}_{xx} + F(t,x), \qquad \widetilde{u}(t,0) = 0, \qquad \widetilde{u}(t,\ell) = 0,$$

$$\widetilde{u}(0,x) = f(x) - \alpha(0) - \frac{\beta(0)-\alpha(0)}{\ell}\,x, \qquad \widetilde{u}_t(0,x) = g(x) - \alpha'(0) - \frac{\beta'(0)-\alpha'(0)}{\ell}\,x,$$

with forcing function

$$F(t,x) = -\,\alpha''(t) - \frac{\beta''(t)-\alpha''(t)}{\ell}\,x.$$

---

9.5.20. (a) $\displaystyle \psi(t,x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\,\mathrm{i}\,\frac{(2k+1)^2\pi^2}{\hbar}\,t\right)\sin(2k+1)\pi x.$

(*b*) Using the Plancherel formula (7.64) and then (3.56), the squared norm is

$$\|\,\psi(t,\cdot\,)\,\|^2 = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = 1.$$

♢ 9.5.25. Dispersion relation: $\omega = k^2/\hbar$; phase velocity: $c_p = k/\hbar$; group velocity: $c_g = 2k/\hbar$.

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# Student Solutions to

## Chapter 10: Finite Elements and Weak Solutions

10.1.1. (a) $Q[u] = \int_0^\pi \left[ \frac{1}{2} u'(x)^2 + (1-x)u(x) \right] dx$; (b) $-u'' = x - 1$, $u(0) = u(\pi) = 0$;

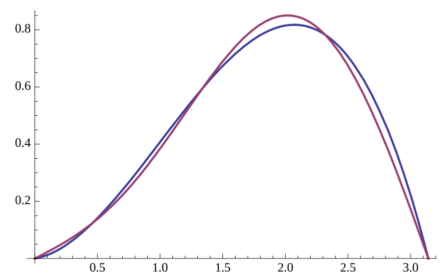
(c) $u_\star(x) = \frac{1}{6}\, x(\pi - x)(x + \pi - 3)$, $Q[u_\star] = -\frac{1}{90}\pi^5 + \frac{1}{24}\pi^4 - \frac{1}{90}\pi^3 \approx -.6334$;

(d) When $w(x) = c_1 \sin x + c_2 \sin 2x$, we have

$$Q[w] = \tfrac{1}{4}\pi c_1^2 + \pi c_2^2 + (2-\pi)c_1 + \tfrac{1}{2}\pi c_2 = P(c_1, c_2);$$

(e) $$w_\star(x) = \left(2 - \frac{4}{\pi}\right)\sin x - \frac{1}{4}\sin 2x, \qquad Q[w_\star] = 4 - \frac{4}{\pi} - \frac{17\pi}{16} \approx -.6112 > Q[u_\star].$$

The maximum deviation between the two is $\| u_\star - w_\star \|_\infty \approx .0680$. In the accompanying plot, $u_\star$ is in blue, and has a smaller maximum than $w_\star$, which is in purple:




♡ 10.1.5. (a) The solution to the corresponding boundary value problem

$$-[(x+1)u']' = 1, \quad u(0) = u(1) = 0, \qquad \text{is} \qquad u_\star(x) = \frac{\log(x+1)}{\log 2} - x;$$

(b) $Q[u] = \int_{-1}^1 \left[\frac{1}{2} u'(x)^2 - (x^2 - x)u(x)\right] dx$ on the space of $\mathrm{C}^2$ functions $u(x)$ satisfying $u(-1) = u(1) = 0$;

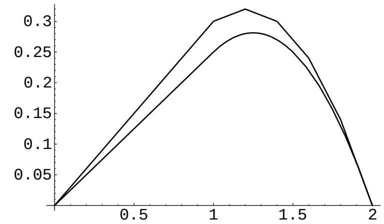
(c) $w_\star(x) = \frac{1}{131}(20x - 55)\, x\,(x-1)$, with $\| u_\star - w_\star \|_2 \approx .00728$, $\| u_\star - w_\star \|_\infty \approx .0011$.

(d) $w_\star(x) = \frac{1}{89}(-7x^2 + 21x - 39)\, x\,(x-1)$, with $\| u_\star - w_\star \|_2 \approx 1.59 \times 10^{-6}$ and $\| u_\star - w_\star \|_\infty \approx 1.38 \times 10^{-4}$. The second approximation has to be at least as good as the first, because every cubic polynomial is an element of the larger quartic subspace: $W_3 \subset W_4$. In fact, it is significantly better.

♠ 10.2.2. (a) Solution:

$$u(x) = \begin{cases} \frac{1}{4}\,x, & 0 \le x \le 1, \\ \frac{1}{4}\,x - \frac{1}{2}\,(x-1)^2, & 1 \le x \le 2; \end{cases}$$

maximal error at sample points: .05;
maximal overall error: .05.




♣ 10.2.4. The solution minimizes the quadratic functional

$$Q[u] = \int_0^1 \left[\, \tfrac{1}{2}\,u'(x)^2 + \tfrac{1}{2}\,(x+1)\,u(x)^2 - e^x\,u(x) \,\right] dx,$$

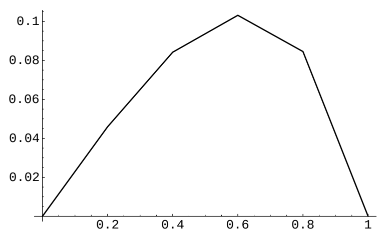
over all functions $u(x)$ that satisfy the boundary conditions. We employ a uniform mesh of step size $h = 1/n$. The finite element matrix entries are given by

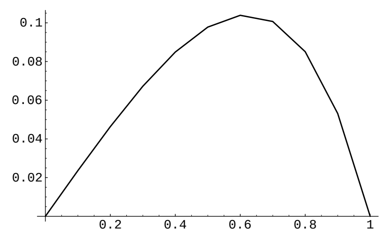
$$k_{ij} = \int_0^1 \left[\, \varphi_i'(x)\,\varphi_j'(x) + (x+1)\,\varphi_i(x)\,\varphi_j(x) \,\right] dx \;\approx\; \begin{cases} \dfrac{2}{h} + \dfrac{2\,h}{3}\,(x_i+1), & i = j, \\[2mm] -\dfrac{1}{h} + \dfrac{h}{6}\,(x_i+1), & |\,i-j\,| = 1, \\[2mm] 0, & \text{otherwise,} \end{cases}$$

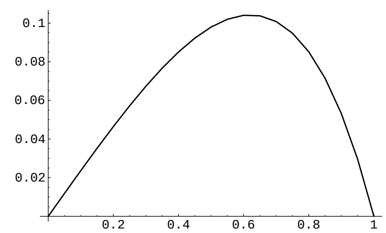
while

$$b_i = \langle\, x\,e^x \,;\varphi_i \,\rangle = \int_0^1 x\,e^x\,\varphi_i(x)\,dx \;\approx\; x_i\,e^{x_i}\,h.$$

Here are the resulting approximations, based on 5, 10, 20 nodes:

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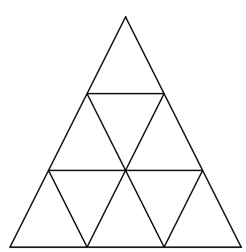
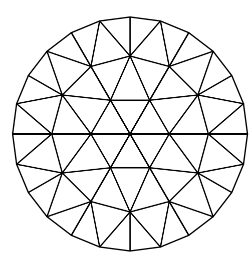
10.3.1. Examples:

$(b)$



$(d)$



10.3.4. (a) $1 - y,\;\; 1 - x,\;\; x + y - 1.$

---

10.3.8. (a) $k_{11} = \frac{5}{2}$, $k_{22} = 1$, $k_{33} = \frac{1}{2}$, $k_{12} = k_{21} = -\frac{3}{2}$, $k_{13} = k_{31} = -1$, $k_{23} = k_{32} = \frac{1}{2}$;

(c) $k_{11} = \frac{1}{2\sqrt{3}} = .288675$, $k_{22} = \frac{\sqrt{3}}{2} = .866025$, $k_{33} = \frac{2}{\sqrt{3}} = 1.154700$,
$k_{12} = k_{21} = 0$, $k_{13} = k_{31} = -\frac{1}{2\sqrt{3}} = -.288675$, $k_{23} = k_{32} = -\frac{\sqrt{3}}{2} = -.866025$.

10.3.10. True — they have the same angles, and so, by (10.46), their stiffnesses will be the same.

---

♠ 10.3.14. (a) $u(x,y) = \dfrac{\sin x \ \sinh(\pi - y)}{\sinh \pi}$, with $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right) = \dfrac{\sinh \frac{1}{2}\pi}{\sinh \pi} \approx .199268$;

(b) $K = (\,4\,)$, $\widetilde{K} = (\,-1 \quad -1 \quad -1 \quad -1\,)$, $\mathbf{b} = \mathbf{0}$, $\mathbf{h} = (\,1,0,0,0\,)^T$.

The solution to (10.59) gives the value $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right) \approx .331812$, with error .132544.

(c) Ordering the interior nodes from left to right, and then bottom to top, and the boundary nodes counterclockwise, starting at the bottom left:

$$K = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix},$$

$$\widetilde{K} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

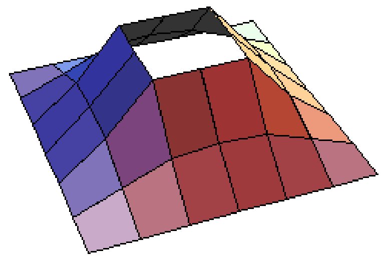
$\mathbf{b} = \mathbf{0}$,

$\mathbf{h} = (\,.707107, 1., .707107, 0, 0, 0, 0, 0, 0, 0, 0, 0\,)^T$.

Now, $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right) \approx .213388$, with error .014120.

(d) $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right) \approx .202915$, with error .003647, so the finite element approximations appear to be converging.

♠ 10.3.18. (a) At the 5 interior nodes on each side of the central square $C$, the computed temperatures are $20.8333,\ 41.6667,\ 45.8333,\ 41.6667,\ 20.8333$:



(b) (i) The minimum temperature on $C$ is 20.8333, achieved at the four corners;

(ii) the maximum temperature is 45.8333, achieved at the four midpoints;

(iii) the temperature is not equal to $50^\circ$ anywhere on $C$.

---

10.4.1. (b) Semi-weak formulation:

$$\int_0^2 \Big[ -e^x u'(x)\, v'(x) + \big( u(x) - e^x u'(x) - \cos x \big)\, v(x) \Big]\, dx = 0$$

for all smooth test functions $v(x) \in \mathrm{C}^1[0,1]$ along with boundary conditions

$$u'(0) = u'(2) = v'(0) = v'(2) = 0.$$

Fully weak formulation:

$$\int_0^2 \Big[ u(x) \big( e^x\, v''(x) + 2\, e^x\, v'(x) + (1 + e^x) v(x) \big) - (\cos x)\, v(x) \Big]\, dx = 0$$

for all $v(x) \in \mathrm{C}^2[0,2]$ along with boundary conditions

$$u'(0) = u'(2) = v'(0) = v'(2) = 0.$$

♢ 10.4.7. Suppose $f(t_0, x_0) > 0$, say. Then, by continuity, $f(t,x) > 0$ for all $x$ in some open ball

$$B_\varepsilon = \left\{ (t,x) \;\middle|\; (t-t_0)^2 + (x-x_0)^2 < \varepsilon^2 \right\}$$

centered at $(t_0, x_0)$. Choose $v(t,x)$ to be a $\mathrm{C}^1$ function that is $> 0$ in $B_\varepsilon$ and $= 0$ outside; for example,

$$v(t,x) = \begin{cases} \big[ (x-x_0)^2 + (t-t_0)^2 - \varepsilon^2 \big]^2, & (t,x) \in B_\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $f(t,x)\, v(t,x) > 0$ inside $B_\varepsilon$ and is $= 0$ everywhere else, which produces the contradiction

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x)\, v(t,x)\, dx\, dt = \iint_{B_\varepsilon} f(t,x)\, v(t,x)\, dt\, dx > 0. \qquad Q.E.D.$$

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# Student Solutions to

## Chapter 11: Dynamics of Planar Media

11.1.1.
$$\frac{\partial u}{\partial t} = \gamma \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \qquad x^2 + y^2 < 1, \quad t > 0,$$
$$\frac{\partial u}{\partial \mathbf{n}} = 0, \qquad x^2 + y^2 = 1, \quad t > 0,$$
$$u(0,x,y) = \sqrt{x^2 + y^2}\,, \qquad x^2 + y^2 < 1.$$
For the Neumann boundary value problem, the equilibrium temperature is the average value of the initial temperature, namely
$$\frac{1}{\pi} \iint_\Omega \sqrt{x^2 + y^2}\; dx\, dy = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta = \frac{2}{3}\,.$$

11.1.3. (a) 0.

♡ 11.1.6. (a) Using (6.85) with $u = 1$ and $v = u$,
$$\frac{dH}{dt} = \iint_\Omega \frac{\partial u}{\partial t}\,(t,x,y)\, dx\, dy = \gamma \iint_\Omega \Delta u\, dx\, dy = \gamma \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\; ds = 0,$$
in view of the homogeneous Neumann boundary conditions. Since its derivative is identically zero, we conclude that $H(t) \equiv H(0)$ is constant.

(b) By part (a),
$$H(t) = H(0) = \iint_\Omega u(0,x,y)\, dx\, dy = T_0 \text{ area } \Omega,$$
where
$$T_0 = \frac{1}{\text{area } \Omega} \iint_\Omega u(0,x,y)\, dx\, dy$$
is the average initial temperature. On the other hand, as $t \to \infty$, the solution approaches a constant equilibrium temperature, $u(t,x,y) \to T_\star$. Thus,
$$T_0 \text{ area } \Omega = \lim_{t\to\infty} H(t) = \iint_\Omega \lim_{t\to\infty} u(t,x,y)\, dx\, dy = \iint_\Omega T_\star\, dx\, dy = T_\star \text{ area } \Omega,$$
and hence $T_\star = T_0$.

11.1.11. (a) Define the linear operator $L[u] = (u_x, u_y, u)^T$, so $L: U \to V$ maps the space of scalar fields $u(x,y)$ satisfying the Neumann boundary conditions to the space of vector-valued functions $\mathbf{v}(x,y) = (v_1(x,y), v_2(x,y), v_3(x,y))^T$ satisfying $(v_1, v_2)^T \cdot \mathbf{n} = 0$ on $\partial\Omega$. Using the $\mathrm{L}^2$ inner product on $U$ and the $\mathrm{L}^2$ inner product
$$\langle\, \mathbf{v}\,;\mathbf{w}\,\rangle = \iint_\Omega v_1(x,y)\, w_1(x,y) + v_2(x,y)\, w_2(x,y) + v_3(x,y)\, w_3(x,y)\, dx\, dy$$
on $V$, the adjoint map $L^*: V \to U$ is given by $L^*[\mathbf{v}] = -\dfrac{\partial v_1}{\partial x} - \dfrac{\partial v_2}{\partial y} + \alpha\, v_3$. Thus, $S[u] = L^* \circ L[u] = -\Delta u + \alpha\, u$, which proves that the evolution equation is in self-adjoint form: $u_t = -S[u]$.

(b) The operator $L$ has trivial kernel, $\ker L = \{0\}$, and so $S = L^* \circ L$ is positive definite. This implies that the boundary value problem $S[u] = 0$ for the equilibrium solution, $-\Delta u + \alpha u = 0$ subject to homogeneous Neumann boundary conditions, has a unique solution, namely $u \equiv 0$.

---

11.2.1. $u(t,x,y) = e^{-2\pi^2 t} \sin \pi x \, \sin \pi y$. The decay is exponential at a rate $2\pi^2$.

11.2.3. $u(t,x,y) = -\dfrac{4}{\pi} \displaystyle\sum_{k=0}^{\infty} e^{-2(2k+1)^2\pi^2 t} \, \frac{\sin(2k+1)\pi(x+1)}{2k+1}$.

♡ 11.2.8. (b) $\dfrac{5\pi^2\gamma}{4a^2}$.

♡ 11.2.12. (a) The equilibrium solution $u_\star(x,y)$ solves the Laplace equation, $u_{xx} + u_{yy} = 0$, subject to the given boundary conditions: $u(0,y) = u(\pi,y) = 0 = u(x,0)$, $u(x,\pi) = f(x)$. Thus,

$$u_\star(x,y) = \sum_{m=1}^{\infty} b_m \sin m x \, \frac{\sinh m y}{\sinh m\pi}, \qquad \text{where} \qquad b_m = \frac{2}{\pi} \int_0^\pi f(x) \sin m x \, dx.$$

(b) $u(t,x,y) = u_\star(x,y) + v(t,x,y)$, so that the "transient" $v(t,x,y)$ solves the initial-boundary value problem

$$v_t = v_{xx} + v_{yy}, \quad v(0,x,y) = -u_\star(x,y), \quad v(0,y) = v(\pi,y) = 0 = v(x,0) = v(x,\pi).$$

Thus, using Exercise 3.2.42, the transient is

$$v(t,x,y) = \frac{2}{\pi} \sum_{m,n=1}^{\infty} \frac{(-1)^n n}{m^2+n^2} \, b_m \, e^{-(m^2+n^2)t} \sin m x \, \sin n y,$$

which decays to zero exponentially fast at a rate of $-2$ provided $b_1 \neq 0$, or, more generally, $-(m^2+1)$ when $b_1 = \cdots = b_{m-1} = 0$, $b_m \neq 0$. Thus, the solution is

$$u(t,x,y) = \sum_{m=1}^{\infty} b_m \sin m x \, \frac{\sinh m y}{\sinh m\pi} + \frac{2}{\pi} \sum_{m,n=1}^{\infty} \frac{(-1)^n n}{m^2+n^2} \, b_m \, e^{-(m^2+n^2)t} \sin m x \, \sin n y.$$

---

11.3.1. (a) $\frac{3}{4}\sqrt{\pi}$.

11.3.5. Use the substitution $x = t^{1/3}$, with $dx = \frac{1}{3} t^{-2/3} \, dt$ to obtain the value

$$\int_0^\infty \sqrt{x} \, e^{-x^3} \, dx = \frac{1}{3} \int_0^\infty e^{-t} \, t^{-1/2} \, dt = \tfrac{1}{3} \Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\sqrt{\pi}.$$

11.3.8. $\widehat{u}(x) = 1 + 2(x-1) + (x-1)^2 = x^2$,

$$\widetilde{u}(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 - \cdots = \sum_{k=0}^{\infty} (-1)^k (x-1)^k = \frac{1}{x}.$$

11.3.10. (a)

$$\widehat{u}(x) = 1 - x^2 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!},$$

$$\begin{aligned}\widetilde{u}(x) &= x - \tfrac{2}{3}x^3 + \tfrac{4}{15}x^5 - \tfrac{8}{105}x^7 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^k x^{2k+1}}{(2k+1)(2k-1)(2k-3)\cdots 5\cdot 3} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} k!\, x^{2k+1}}{(2k+1)!}.\end{aligned}$$

(*b*) Since the equation has no singular points, both series have an infinite radius of convergence. (*c*) $\widehat{u}(x) = e^{-x^2}$; (*d*) $\widetilde{u}(x) = e^{-x^2}\int_0^x e^{y^2}\,dy$.

11.3.15. (a) $u(x) = c_1\widehat{u}(c-x) + c_2\widetilde{u}(c-x)$, where $\widehat{u}, \widetilde{u}$ are given in (11.81, 82), and $c_1, c_2$ are arbitrary constants.

---

11.3.22. (a) Multiplying by $x$, the equation $2x^2u'' + xu' + x^2u = 0$ has the form (11.88) at $x_0 = 0$ with $a(x) = 2$, $b(x) = 1$, $c(x) = x^2$, all analytic at $x = 0$ with $a(0) \neq 0$.

(*b*) The indicial equation is $2r^2 - r = 0$ with roots $r = 0, \frac{1}{2}$. The recurrence formula is

$$u_n = -\frac{u_{n-2}}{(n+r)[2(n+r)-1]}, \qquad n \geq 2.$$

The resulting two solutions are

$$\begin{aligned}\widehat{u}(x) = 1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{(2\cdot 4)\cdot(3\cdot 7)} - \frac{x^6}{(2\cdot 4\cdot 6)\cdot(3\cdot 7\cdot 11)} \\ + \cdots + \frac{(-1)^k x^{2k}}{\big(2\cdot 4\cdots(2k)\big)\cdot\big(3\cdot 7\cdots(4k-1)\big)} + \cdots,\end{aligned}$$

$$\begin{aligned}\widetilde{u}(x) = x^{1/2} - \frac{x^{5/2}}{2\cdot 5} + \frac{x^{9/2}}{(2\cdot 4)\cdot(5\cdot 9)} - \frac{x^{13/2}}{(2\cdot 4\cdot 6)\cdot(5\cdot 9\cdot 13)} \\ + \cdots + \frac{(-1)^k x^{(4k+1)/2}}{\big(2\cdot 4\cdots(2k)\big)\cdot\big(5\cdot 9\cdots(4k+1)\big)} + \cdots.\end{aligned}$$

11.3.27. (a) $J_{5/2}(x) = \sqrt{\dfrac{2}{\pi}}\left(3x^{-5/2}\sin x - 3x^{-3/2}\cos x - x^{-1/2}\sin x\right)$.

♢ 11.3.30. (a) The point $x_0 = 0$ is regular because it is of the form (11.88) with

$$a(x) = b(x) = 1, \quad c(x) = -x^2 + m^2.$$

(*b*) Replacing $x \mapsto \mathrm{i}\,x$ converts (11.114) to the ordinary Bessel equation (11.98) of order $m$. Therefore, its Frobenius solution(s) are

$$\widehat{u}(x) = J_m(\mathrm{i}\,x) = \mathrm{i}^m \sum_{k=0}^{\infty} \frac{x^{m+2k}}{2^{2k+m}\, k!\, \Gamma(m+k+1)},$$

and, if $m$ is not an integer,

$$\widetilde{u}(x) = J_{-m}(\mathrm{i}\,x) = \mathrm{i}^{-m} \sum_{k=0}^{\infty} \frac{(-1)^k x^{-m+2k}}{2^{2k-m}\, k!\, \Gamma(-m+k+1)}.$$

If $m$ is an integer, then the second solution is $u_2(x) = Y_m(\mathrm{i}\,x)$, where $Y_m$ is the Bessel function of the second kind (11.107).

11.4.1. $$u(t,r,\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{J_0\big(\frac{1}{2}\zeta_{0,n}\big)}{J_1(\zeta_{0,n})^2} e^{-\zeta_{0,n}^2 t} J_0(\zeta_{0,n} r) + {} + \frac{2}{\pi} \sum_{m,n=1}^{\infty} \frac{J_m\big(\frac{1}{2}\zeta_{m,n}\big)}{J_{m+1}(\zeta_{m,n})^2} e^{-\zeta_{m,n}^2 t} J_m(\zeta_{m,n} r) \cos m\theta.$$

♢ 11.4.7. Suppose $u(t,x,y)$ solves the heat equation $u_t = \gamma\,\Delta u$ on a disk of radius 1, subject to initial conditions $u(0,x,y) = f(x,y)$ and, say, homogeneous boundary conditions. Then $U(t,x,y) = u(t/R^2, x/R, y/R)$ solves the heat equation $U_t = \gamma\,\Delta U$ on a disk of radius $R$, subject to initial conditions $U(0,x,y) = F(x,y)$ where $f(x,y) = F(x/R, y/R)$, along with the same type of homogeneous boundary conditions.

11.4.9. 12 minutes.

♡ 11.4.15. (a) The eigensolutions are

$$u_{m,n}(t,r,\theta) = e^{-\zeta_{m,n}^2 t} J_m(\zeta_{m,n} r) \sin m\theta, \qquad \begin{array}{l} n = 1,2,3,\ldots, \\ m = 1,2,\ldots. \end{array}$$

The general solution is a series in the eigensolutions:

$$u(t,r,\theta) = \sum_{m,n=1}^{\infty} b_{m,n}\, u_{m,n}(t,r,\theta),$$

whose coefficients $b_{m,n}$ are prescribed by the initial data.

(b) The eigensolutions are

$$\begin{aligned} u_{0,n}(t,r) &= e^{-\zeta_{0,n}^2 t} J_0(\zeta_{0,n} r), && n = 1,2,3,\ldots, \\ u_{m,n}(t,r,\theta) &= e^{-\zeta_{m,n}^2 t} J_m(\zeta_{m,n} r) \cos m\theta, && m = 1,2,\ldots. \end{aligned}$$

where $\eta_{m,n}$ are roots of the *derivative* of the $m^{\text{th}}$ order Bessel function: $J_m'(\eta_{m,n}) = 0$. The general solution is a series in the eigensolutions:

$$u(t,r,\theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_{0,n}\, u_{0,n}(t,r) + \sum_{m,n=1}^{\infty} a_{m,n}\, u_{m,n}(t,r,\theta),$$

whose coefficients $b_{m,n}$ are prescribed by the initial data.

(c) The Dirichlet problem decays to equilibrium over 2.5 times faster than the mixed boundary value problem. For the Dirichlet problem, the decay rate is $\zeta_{1,1}^2 \approx 14.682$, whereas for the mixed problem, the rate is $\zeta_{0,1}^2 \approx 5.783$. Intuitively, the greater the portion of the boundary that is held fixed at $0^\circ$, the faster the return to equilibrium.

11.4.19. In view of the formula (11.105) for $J_{1/2}(x)$, the roots are $\zeta_{1/2,n} = n\pi$ for $k = 1,2,3,\ldots$. In this case they exactly satisfy (11.119).

11.5.1. $$u(t,x,y) = \frac{1}{20t+1} e^{-(x^2+y^2)/(20t+1)}.$$

♢ 11.5.6. $$u(t,x,y) = \int_0^t \left( \iint \frac{h(\tau,\xi,\eta)}{4\pi\gamma(t-\tau)} e^{-[(x-\xi)^2+(y-\eta)^2]/[4\gamma(t-\tau)]}\, d\xi\, d\eta \right) d\tau.$$

11.5.11. (a) Since $v(t,x)$ solves $v_t = -\,v_{xxx}$, while $w(t,y)$ solves $w_t = -\,w_{yyy}$, we have
$$u_t = v_t w + v\,w_t = -\,v_{xxx} w - v\,w_{yyy} = -\,u_{xxx} - u_{yyy}.$$
(b) $\displaystyle F(t,x,y;\xi,\eta) = \frac{1}{(3t)^{2/3}}\,\mathrm{Ai}\left(\frac{x-\xi}{\sqrt[3]{3t}}\right)\mathrm{Ai}\left(\frac{y-\eta}{\sqrt[3]{3t}}\right).$

(c) $\displaystyle u(t,x,y) = \frac{1}{(3t)^{2/3}}\iint f(\xi,\eta)\,\mathrm{Ai}\left(\frac{x-\xi}{\sqrt[3]{3t}}\right)\mathrm{Ai}\left(\frac{y-\eta}{\sqrt[3]{3t}}\right)d\xi\,d\eta.$

---

11.6.3. $\omega_{1,1} = \sqrt{2}\,\pi \approx 4.4429$; two independent normal modes;
$\omega_{1,2} = \omega_{2,1} = \sqrt{5}\,\pi \approx 7.0248$; four independent normal modes;
$\omega_{2,2} = 2\sqrt{2}\,\pi \approx 8.8858$; two independent normal modes;
$\omega_{1,3} = \omega_{3,1} = \sqrt{10}\,\pi \approx 9.9346$; four independent normal modes;
$\omega_{2,3} = \omega_{3,2} = \sqrt{13}\,\pi \approx 11.3272$; four independent normal modes;
$\omega_{1,4} = \omega_{4,1} = \sqrt{17}\,\pi \approx 12.9531$; four independent normal modes.

11.6.5. (a)
$$u(t,x,y) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos\sqrt{1+\left(k+\frac12\right)^2}\;\pi t\;\sin\left(k+\frac12\right)\pi x\;\sin\pi y}{k+\frac12} +$$
$$+\;\frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{4\sin\sqrt{1+\left(k+\frac12\right)^2}\;\pi t\;\sin\left(k+\frac12\right)\pi x\;\sin\pi y}{\left(k+\frac12\right)\sqrt{1+\left(k+\frac12\right)^2}}\,;$$
(b) $u(t,x,y) = \cos\pi t\,\sin\pi y + \dfrac{1}{\pi}\sin\pi t\,\sin\pi y.$

11.6.9. For example, $u(t,x,y) = c_1\cos\sqrt{2}\,\pi\,c\,t\;\sin\pi x\;\sin\pi y + c_2\cos 2\sqrt{2}\,\pi\,c\,t\;\sin 2\pi x\;\sin 2\pi y$, for any $c_1, c_2 \neq 0$ is periodic of period $1/\sqrt{2}$, but a linear combination of two fundamental modes. Its frequency, $2\sqrt{2}\,\pi$, is (necessarily) a fundamental frequency.

11.6.14. $u(t,x,y) = \dfrac{2}{c}\sin(ct)\,J_0\left(\sqrt{x^2+y^2}\,\right)$. The vibrations are radially symmetric and periodic with period $2\pi/c$. For fixed $t$, the solution is either identically 0 or of one sign throughout the interior of the disk. Thus, at any given time, the drum is either entirely above the $(x,y)$–plane, entirely below it, or, momentarily, completely flat.

11.6.18. (a) The displacement $u(t,r,\theta)$ must satisfy $u_{tt} = c^2\Delta u$, along with the boundary and initial conditions
$$u(t,r,0) = 0, \qquad u(t,1,\theta) = 0, \qquad u_\theta\left(t,r,\tfrac12\pi\right) = 0, \qquad r < 1,$$
$$u(t,r,\theta) = 0, \qquad u_t(0,r,\theta) = 2\,\delta\left(r-\tfrac12\right)\delta\left(\theta-\tfrac14\pi\right), \qquad 0 < \theta < \tfrac12\pi.$$
Note that the factor of 2 in the initial condition for $u_t$ comes from the formula
$$\delta(x-x_0,y-y_0) = \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)}{r} = \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)}{r_0} \qquad \text{for} \qquad r_0 \neq 0,$$
relating the rectangular and polar coordinate forms of the delta function.

(b) The odd-order Bessel roots $\zeta_{2k+1,n}$ for $k = 0,1,2,\ldots\,,\ n = 1,2,3,\ldots\,.$

$(c)$ $u(t,r,\theta) =$

$$\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{8\,J_{2k+1}\big(\tfrac12\,\zeta_{2k+1,n}\big)\sin\big(\tfrac12 k+\tfrac14\big)\pi}{\pi\,\zeta_{2k+1,n}\,J_{2k+2}(\zeta_{2k+1,n})^2}\ \sin(\zeta_{2k+1,n}\,t)\,J_{2k+1}(\zeta_{2k+1,n}\,r)\,\sin(2k+1)\theta.$$

$(d)$ The motion of the quarter disk is stable and quasiperiodic.

---

11.6.25. If the side lengths are $a \le b$, then the two lowest vibrational frequencies are

$$\omega_1 = \pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} \ < \ \omega_2 = \pi\sqrt{\frac{1}{a^2}+\frac{4}{b^2}}\,.$$

Thus, we can recover the side lengths through the formulae

$$a = \pi\sqrt{\frac{3}{4\,\omega_1^2-\omega_2^2}}\,, \qquad b = \pi\sqrt{\frac{3}{\omega_2^2-\omega_1^2}}\,.$$

11.6.29. Since the half disk's vibrational frequencies are a subset of the full disk frequencies, corresponding to the eigenfunctions $v(x,y)$ that are odd in $y$ — i.e., those, as in (11.156), that involve $\sin m\,\theta$ — the ratios of two half disk frequencies is a ratio of the corresponding full disk frequencies. However, the lowest frequency of the half disk is not the lowest frequency of the full disk, and so the relative frequencies are different.

11.6.34. Set $\xi = a\,x+b\,y$. Then, by the chain rule, $u_{tt} = v_{tt}$, and $u_{xx}+u_{yy} = (a^2+b^2)v_{\xi\xi}$, and hence $v(t,\xi)$ satisfies the wave equation $v_{tt} = c^2 v_{\xi\xi}$ with wave speed $c = 1/\sqrt{a^2+b^2}$ . The solutions are plane waves that have the same value along each line $a\,x+b\,y$ =constant, and move with speed $c$ in the transverse direction.

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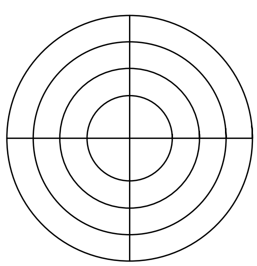
11.6.40. The nodal circle in the fourth mode, with frequency $\omega_{0,2} = 2.29542$, has radius $\zeta_{0,1}/\zeta_{0,2} \approx .43565$; in the sixth mode, with frequency $\omega_{1,2} = 2.9173$, the radius is $\zeta_{1,1}/\zeta_{1,2} \approx .54617$; in the eighth mode, with frequency $\omega_{2,2} = 3.50015$, the radius is $\zeta_{2,1}/\zeta_{2,2} \approx .61013$; in the ninth mode, with frequency $\omega_{0,3} = 3.59848$, the two radii are $\zeta_{0,1}/\zeta_{0,3} \approx .27789$ and $\zeta_{0,2}/\zeta_{0,3} \approx .63788$. Thus,

$$\zeta_{0,1}/\zeta_{0,3} < \zeta_{0,1}/\zeta_{0,2} < \zeta_{1,1}/\zeta_{1,2} < \zeta_{2,1}/\zeta_{2,2} < \zeta_{0,2}/\zeta_{0,3}.$$

11.6.41. $(b)$



radii: .3471, .5689, .7853;

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# Student Solutions to

## Chapter 12: Partial Differential Equations in Space

12.1.1. For example: (a) $1,\ x,\ y,\ z,\ x^2-y^2,\ y^2-z^2,\ x\,y,\ x\,z,\ y\,z.$

♢ 12.1.4. By the chain rule, $U_{xx}=u_{xx},\ \ U_{yy}=u_{yy},\ \ U_{zz}=u_{zz}$, and hence $\Delta U=\Delta u=0$.

♢ 12.1.10. (a)
$$\begin{aligned}\nabla\cdot(u\,\mathbf{v}) &= \frac{\partial}{\partial x}\,(u\,v_1)+\frac{\partial}{\partial y}\,(u\,v_2)+\frac{\partial}{\partial z}\,(u\,v_3)\\ &= \bigl(\,u_x\,v_1+u_y\,v_2+u_z\,v_3\,\bigr)+u(v_{1,x}+v_{2,y}+v_{3,z})=\nabla u\cdot\mathbf{v}+u\,\nabla\cdot\mathbf{v};\end{aligned}$$

♢ 12.1.11. First, setting $\mathbf{v}=\nabla v$ in (12.10) produces
$$\iiint_\Omega u\,\Delta v\,dx\,dy\,dz=\iint_{\partial\Omega} u\,\frac{\partial v}{\partial\mathbf{n}}\,dS-\iiint_\Omega \nabla u\cdot\nabla v\,dx\,dy\,dz.$$
Taking $v=1$ in the latter identity yields (*a*), while setting $v=u$ gives (*b*).

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12.2.2. The equilibrium temperature is constant: $u\equiv 10^\circ$.

12.2.4. (*b*) $u(x,y,z)=\frac{2}{3}+\frac{1}{3}\,x^2+\frac{1}{3}\,y^2-\frac{2}{3}\,z^2.$

12.2.5. (*i*) $\frac{1}{2}\,\pi=\dfrac{1}{4\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^{\pi}\varphi\sin\varphi\ d\varphi\,d\theta;\qquad$ (*ii*) $\frac{1}{2}\,\pi-\frac{3}{8}\,\pi\,z+\frac{21}{256}\,\pi\,(x^2+y^2)\,z-\frac{7}{128}\,\pi\,z^3.$

♢ 12.2.10. (a) At $t=1$ we write (12.28) as
$$(t-1)^2(t+1)\,\frac{d^2P}{dt^2}+(t-1)(2\,t)\,\frac{dP}{dt}-\mu\,(t-1)\,P=0,$$
which is of the form (11.88) with
$$p(t)=(t+1),\qquad q(t)=2\,t,\qquad r(t)=-\,\mu\,(t-1),$$
all analytic at $t=1$ with $p(1)=2\neq 0$. A similar argument applies at $t=-1$.

(*b*) Set
$$K[\,u(t)\,]=(1-t^2)\,u''(t)-2\,t\,u'(t)+\mu\,u(t)=d/dt\,\bigl[\,(1-t^2)\,u'(t)\,\bigr]+\mu\,u(t).$$
Then, integrating by parts twice,
$$\begin{aligned}\langle\,K[\,u\,]\,;v\,\rangle &= \int_{-1}^{1}\left[\,\frac{d}{dt}\bigl[\,(1-t^2)\,u'(t)\,\bigr]+\mu\,u(t)\,\right]v(t)\,dt\\ &= (1-t^2)\,u'(t)\,v(t)\,\Big|_{t=-1}^{1}+\int_{-1}^{1}\bigl[-(1-t^2)\,u'(t)v'(t)+\mu\,u(t)\,v(t)\,\bigr]\,dt\\ &= (1-t^2)\,\bigl[\,u'(t)\,v(t)-u(t)\,v'(t)\,\bigr]\,\Big|_{t=-1}^{1}+\int_{-1}^{1}u(t)\left[\,\frac{d}{dt}\,(1-t^2)\,v'(t)+\mu\,v(t)\,\right]dt\\ &= \langle\,u\,;K[\,v\,]\,\rangle,\end{aligned}$$
where the boundary terms vanish at $\pm1$ provided $u(\pm1),u'(\pm1),v(\pm1),v'(\pm1)$ are all finite.

(*c*) The Legendre polynomials $P_k(t)$ are the eigenfunctions of the self-adjoint operator
$$K_0[\,u(t)\,]=(1-t^2)\,u''(t)-2\,t\,u'(t)$$

corresponding to the eigenvalues $\lambda_k = k\,(k+1)$. This implies that they are orthogonal with respect to the $\mathrm{L}^2$ inner product on $[-1,1]$, i.e.,

$$\langle\, P_k\,; P_l\,\rangle = \int_{-1}^{1} P_k(t)\,P_l(t)\,dt = 0 \qquad \text{for} \qquad k \neq l.$$

♢ 12.2.15. Using (12.31)

$$\begin{aligned}\sqrt{1-t^2}\;\frac{dP_n^m}{dt} + \frac{m\,t}{\sqrt{1-t^2}}\,P_n^m(t) &= (1-t^2)^{(m+1)/2}\,\frac{d^{m+1}}{dt^{m+1}}\,P_n(t)\\ &\quad -m\,t\,(1-t^2)^{(m-1)/2}\,\frac{d^m}{dt^m}\,P_n(t) + m\,t\,(1-t^2)^{(m-1)/2}\,\frac{d^m}{dt^m}\,P_n(t) = P_n^{m+1}(t).\end{aligned}$$

12.2.19. Since $Y_0^0(\varphi,\theta) = 1$, the first surface is the unit sphere $r = 1$. The surface $r = Y_1^0(\varphi,\theta) = \cos\varphi$ can be rewritten as $r^2 = r\cos\varphi = z$, or, equivalently, $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$, which is the sphere of radius $\frac{1}{2}$ centered at the point $\left(0,0,\frac{1}{2}\right)$.

♢ 12.2.23. According to (12.46),

$$\begin{aligned}\langle\,\mathcal{Y}_n^m\,;\mathcal{Y}_l^k\,\rangle &= \langle\, Y_n^m + \mathrm{i}\,\widetilde{Y}_n^m\,; Y_l^k + \mathrm{i}\,\widetilde{Y}_l^k\,\rangle\\ &= \langle\, Y_n^m\,; Y_l^k\,\rangle + \mathrm{i}\,\langle\,\widetilde{Y}_n^m\,; Y_l^k\,\rangle - \mathrm{i}\,\langle\, Y_n^m\,;\widetilde{Y}_l^k\,\rangle + \langle\,\widetilde{Y}_n^m\,;\widetilde{Y}_l^k\,\rangle = \langle\, Y_n^m\,; Y_l^k\,\rangle + \langle\,\widetilde{Y}_n^m\,;\widetilde{Y}_l^k\,\rangle,\end{aligned}$$

by the orthogonality of the real spherical harmonics — which continues to apply when $m$ and/or $k$ is negative in view of our conventions that $Y_n^m = Y_n^{-m}$, $\widetilde{Y}_n^m = -\widetilde{Y}_n^{-m}$, $\widetilde{Y}_n^0 \equiv 0$. Thus, if $(m,n) \neq (k,l)$ both of the final summands are zero, proving orthogonality. On the other hand, we find

$$\|\,\mathcal{Y}_n^m\,\|^2 = \|\,Y_n^m\,\|^2 + \|\,\widetilde{Y}_n^m\,\|^2 = \frac{4\pi(n+m)!}{(2n+1)(n-m)!}\,,$$

since, when $m \neq 0$ the two norms are equal by the second formula in (12.42), whereas when $m = 0$, the second norm is zero by our convention, and the first formula in (12.42) applies.

12.2.26. (a) (i) $\frac{9}{64}\,r^4 + \frac{5}{16}\,r^4\cos 2\varphi + \frac{35}{64}\,r^4\cos 4\varphi$; (ii) $\frac{3}{8}\,x^4 + \frac{3}{4}\,x^2y^2 + \frac{3}{8}\,y^4 - 3\,x^2z^2 - 3\,y^2z^2 + z^4$.

12.2.29. (a) $K_0^0(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,.$

12.2.33. (a) $\langle\, f\,; g\,\rangle = \displaystyle\int_0^1\!\int_{-\pi}^{\pi}\!\int_0^{\pi} f(r,\theta,\varphi)\,g(r,\theta,\varphi)\,r^2\sin\varphi\;d\varphi\,d\theta\,dr,$

$$\|\,f\,\| = \sqrt{\int_0^1\!\int_{-\pi}^{\pi}\!\int_0^{\pi} f(r,\theta,\varphi)^2\,r^2\sin\varphi\;d\varphi\,d\theta\,dr}\;.$$

(b) Since $f(r,\varphi,\theta) = r\cos\varphi$, $g(r,\varphi,\theta) = r^2\sin^2\varphi$,

$$\|\,f\,\| = \sqrt{\tfrac{4}{15}\,\pi} \approx .9153, \qquad \|\,g\,\| = \sqrt{\tfrac{32}{105}\,\pi} \approx .9785, \qquad \langle\, f\,; g\,\rangle = 0.$$

(c) $|\,\langle\, f\,; g\,\rangle\,| = 0 \leq .8956 = \|\,f\,\|\,\|\,g\,\|$, $\quad \|\,f+g\,\| = \sqrt{\frac{4}{7}\,\pi} \approx 1.3398 \leq 1.8938 = \|\,f\,\| + \|\,g\,\|$.

12.2.35. $u(x,y,z) = 100\,z$.

---

12.3.1. (a) $u(x,y,z) = \dfrac{1}{4\pi\sqrt{x^2+y^2+z^2}} - \dfrac{1}{4\pi}\,.$

12.3.6. In terms of $r = \| \mathbf{x} \|$, the potential is $u(r) = \begin{cases} \dfrac{\rho^2}{2} - \dfrac{\rho^3}{3} - \dfrac{r^2}{6}, & r \le \rho, \\ \dfrac{\rho^3}{3}\left(\dfrac{1}{r} - 1\right), & \rho \le r \le 1. \end{cases}$

12.3.9. $K_0^0(x,y,z) = \dfrac{1}{r} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ is the Newtonian potential, while $\widetilde{K}_0^0(x,y,z) = 0$.

---

♦ 12.4.1. (a) $u_t = \gamma\,\Delta u$, $u(0,x,y,z) = f(x,y,z)$, $u(t,0,y,z) = u(t,a,y,z) = u(t,x,0,z) = u(t,x,b,z) = u(t,x,y,0) = u(t,x,y,c) = 0$, for $(x,y,z) \in B$ and $t > 0$.

(b) For $j,k,l = 1,2,\ldots$.
$$u_{j,k,l}(t,x,y,z) = \exp\left[ -\left( \frac{j^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} \right) \pi^2 \gamma\, t \right] \sin\frac{j\pi x}{a}\ \sin\frac{k\pi y}{b}\ \sin\frac{l\pi z}{c}\,.$$

(c) $u(t,x,y,z) = \displaystyle\sum_{j,k,l=1}^{\infty} c_{j,k,l}\, u_{j,k,l}(t,x,y,z),$ where
$$c_{j,k,l} = \frac{8}{a\,b\,c}\int_0^c\int_0^b\int_0^a f(x,y)\ \sin\frac{j\pi x}{a}\ \sin\frac{k\pi y}{b}\ \sin\frac{l\pi z}{c}\,dx\,dy\,dz.$$

(d) The equilibrium temperature is $u_\star \equiv 0$. The exponential decay rate for most initial data is given by the smallest positive eigenvalue, namely $\lambda_{1,1,1} = \gamma\,\pi^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}\right)$.

♦ 12.4.5. (a) $\dfrac{\partial u}{\partial t} = \gamma\left( \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2} \right)$, $u(t,a,\theta,z) = u(t,r,\theta,0) = u(t,r,\theta,b) = 0$, $u(t,r,-\pi,z) = u(t,r,\pi,z)$, $\dfrac{\partial u}{\partial \theta}(t,r,-\pi,z) = \dfrac{\partial u}{\partial \theta}(t,r,\pi,z)$, $u(0,r,\theta,z) = f(r,\theta,z)$, for $0 < r < a$, $-\pi < \theta < \pi$, $0 < z < h$.

(b) The separable solutions are
$$u_{m,n,k}(t,r,\theta,z) = e^{-\lambda_{m,n,k}t}\, J_m\!\left(\frac{\zeta_{m,n}\,r}{a}\right)\cos m\theta\ \sin\frac{k\pi z}{h}\,, \qquad m = 0,1,2,\ldots\,,$$
$$\widehat{u}_{m,n,k}(r,\theta,z) = e^{-\lambda_{m,n,k}t}\, J_m\!\left(\frac{\zeta_{m,n}\,r}{a}\right)\sin m\theta\ \sin\frac{k\pi z}{h}\,, \qquad n,k = 1,2,3,\ldots\,,$$
where the eigenvalues are
$$\lambda_{m,n,k} = \gamma\left( \frac{\zeta_{m,n}^2}{a^2} + \frac{k^2\pi^2}{h^2} \right).$$
The solution can be written as a Fourier–Bessel series
$$u(t,r,\theta,z) = \frac{1}{2}\sum_{n,k=1}^{\infty} a_{0,n,k}\,u_{0,n,k}(t,r,z) + \sum_{m,n,k=1}^{\infty}\left[ a_{m,n,k}\,u_{m,n,k}(t,r,\theta,z) + b_{m,n,k}\,\widehat{u}_{m,n,k}(t,r,\theta,z) \right],$$
where
$$a_{m,n,k} = \frac{4}{\pi\,a^2\,h\,J_{m+1}(\zeta_{m,n})^2}\int_0^h\int_{-\pi}^{\pi}\int_0^a f(r,\theta,z)\,J_m\!\left(\frac{\zeta_{m,n}\,r}{a}\right) r\cos m\theta\ \sin\frac{k\pi z}{h}\,dr\,d\theta\,dz,$$
$$b_{m,n,k} = \frac{4}{\pi\,a^2\,h\,J_{m+1}(\zeta_{m,n})^2}\int_0^h\int_{-\pi}^{\pi}\int_0^a f(r,\theta,z)\,J_m\!\left(\frac{\zeta_{m,n}\,r}{a}\right) r\sin m\theta\ \sin\frac{k\pi z}{h}\,dr\,d\theta\,dz.$$

($c$) The equilibrium temperature is $u_\star(r,\theta,z) = 0$.

($d$) The (exponential) decay rate is governed by the smallest eigenvalue, namely

$$\lambda_{0,1,1} = \gamma \left( \frac{\zeta_{0,1}^2}{a^2} + \frac{\pi^2}{h^2} \right).$$

---

12.4.14. $15 \times (300/200)^{2/3} = 19.66$ minutes.

♢ 12.4.16. The decay rate is the smallest positive eigenvalue of the Helmholtz boundary value problem $\gamma\,\Delta v + \lambda v = 0$ on the ball of radius $R$ with $v = 0$ on its boundary. The rescaled function $V(\mathbf{x}) = v(\mathbf{x}/R)$ solves the rescaled boundary value problem $\Delta V + \Lambda V = 0$ on the unit ball, $V = 0$ on its boundary, with $\Lambda = R^2\lambda/\gamma = \pi^2$ the smallest eigenvalue. Thus, the decay rate is $\lambda = \pi^2\,\gamma/R^2$.

12.4.20. We can assume, by rescaling, that the common volume is 1, and that the thermal diffusivity is $\gamma = 1$. Note that we are dealing with homogeneous Dirichlet boundary conditions. For the cube, the smallest eigenvalue is $3\pi^2 \approx 29.6088$. For the sphere of radius $R = \sqrt[3]{\dfrac{3}{4\pi}}$, of unit volume, the smallest eigenvalue is $\dfrac{\pi^2}{R^2} = \dfrac{2^{4/3}\pi^{8/3}}{3^{2/3}} \approx 25.6463$, cf. Exercise 12.4.16. Thus, the cube cools down faster.

12.4.25. By l'Hôpital's rule: ($a$) $S_1(0) = \displaystyle\lim_{x\to 0} S_1(x) = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2} = \lim_{x\to 0} \frac{x\sin x}{2x} = 0.$

---

12.4.31. (a)

$$u(t,x,y,z) = \frac{1}{8\,(\pi\,t)^{3/2}} \iiint_{\xi^2+\eta^2+\zeta^2\le 1} 100\, e^{-[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]/(4\,t)}\, d\xi\, d\eta\, d\zeta$$

$$= \frac{12.5}{(\pi\,t)^{3/2}} \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 e^{-[(x-r\sin\varphi\,\cos\theta)^2+(y-r\sin\varphi\,\sin\theta)^2+(z-r\cos\varphi)^2]/(4\,t)}\, r^2\sin\varphi\; dr\, d\varphi\, d\theta.$$

($b$) Since the temperature only depends on the radial coordinate $r$, we set $x = 0$, $y = 0$, $z = \rho$, to simplify the integral:

$$\frac{12.5}{(\pi\,t)^{3/2}} \int_0^1\int_0^{\pi}\int_{-\pi}^{\pi} e^{-(r^2-2\,r\,\rho\cos\varphi+\rho^2)/(4\,t)}\, r^2\sin\varphi\; d\theta\, d\varphi\, dr$$

$$= \frac{50}{\sqrt{\pi\,t}\,\rho} \int_0^1 r\left[ e^{-(r-\rho)^2/(4\,t)} - e^{-(r+\rho)^2/(4\,t)} \right] dr$$

$$= 50\,\mathrm{erf}\left(\frac{\rho+1}{2\sqrt{t}}\right) - 50\,\mathrm{erf}\left(\frac{\rho-1}{2\sqrt{t}}\right) + \frac{100}{\rho}\sqrt{\frac{t}{\pi}}\left[ e^{-(\rho+1)^2/(4\,t)} - e^{-(\rho-1)^2/(4\,t)} \right],$$

which gives the solution value $u(t,x,y,z)$ when $\rho = \sqrt{x^2+y^2+z^2}$.

12.4.36. As in (9.128),

$$F(t,x,y,z;\xi,\eta,\zeta) = 8 \sum_{j,k,l=1}^{\infty} e^{-(j^2+k^2+l^2)\,t} \sin j\,\pi x\, \sin k\pi y\, \sin l\,\pi z\, \sin j\,\pi\xi\, \sin k\pi\eta\, \sin l\,\zeta.$$

---

12.5.1. (a) $u(t,x,y,z) =$

$$\frac{64}{\pi^3}\sum_{i,j,k=0}^{\infty}\frac{\cos\pi\sqrt{(2i+1)^2+(2j+1)^2+(2k+1)^2}\;t\;\sin(2i+1)\pi x\;\sin(2j+1)\pi y\;\sin(2k+1)\pi z}{(2i+1)(2j+1)(2k+1)};$$

(d) $u(t,x,y,z) = \cos 3\pi\, t\, \sin 3\pi x + \dfrac{\sin 2\pi t\, \sin 2\pi y}{2\pi}$.

12.5.3. (a) Assuming the cube is given by $\{\,0 \le x,y,z \le 1\,\}$, the separable eigenmodes are

$$\cos c\sqrt{l^2+m^2+n^2}\;\pi t\;\sin l\pi x\;\sin m\pi y\;\sin n\pi z,$$
$$\sin c\sqrt{l^2+m^2+n^2}\;\pi t\;\sin l\pi x\;\sin m\pi y\;\sin n\pi z,$$

for $l,m,n$ positive integers.

(b) Whenever $l^2+m^2+n^2 = \widehat{l}^{\,2}+\widehat{m}^2+\widehat{n}^2$, one can take a linear combination of the separable modes that is periodic with frequency $\omega = c\sqrt{l^2+m^2+n^2}\;\pi$. For example,

$$u(t,x,y,z) = \cos\sqrt{6}\,c\pi\,t\;(\sin\pi x\;\sin\pi y\;\sin 2\pi z + \sin\pi x\;\sin 2\pi y\;\sin\pi z).$$

12.5.6. (b) The eigenfunctions for the Laplacian operator on the cylinder are

$$u_{m,n,k}(t,r,\theta,z) = J_m(\zeta_{m,n}\,r)\;\cos m\theta\;\;\cos\tfrac{1}{2}\,k\pi z, \qquad m,k = 0,1,2,\ldots,$$
$$\widehat{u}_{m,n,k}(r,\theta,z) = J_m(\zeta_{m,n}\,r)\;\sin m\theta\;\;\cos\tfrac{1}{2}\,k\pi z, \qquad n = 1,2,3,\ldots,$$

with associated eigenvalues and vibrational frequencies

$$\lambda_{m,n,k} = \zeta_{m,n}^2 + \tfrac{1}{4}\,k^2\pi^2, \qquad \omega_{m,n,k} = \sqrt{\lambda_{m,n,k}} = \sqrt{\zeta_{m,n}^2 + \tfrac{1}{4}\,k^2\pi^2}\,.$$

12.5.10. For the sphere, the slowest vibrational frequency is $c\pi/R \approx 3.1416\,c/R$, whereas for the disk it is $c\,\zeta_{0,1}/R \approx 2.4048\,c/R$. Thus, the sphere vibrates faster.

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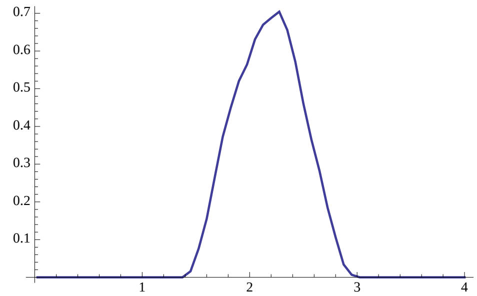
12.6.1. (a) $u(t,x,y,z) = x + z$.

12.6.3. (a) The solution, for $t > 0$, is

$$u(t,\mathbf{x}) = \frac{1}{4\pi t}\iint_{\|\,\boldsymbol{\xi}-\mathbf{x}\,\|=t}\frac{\partial u}{\partial t}\,(0,\mathbf{x})\,dS = \frac{1}{4\pi t}\,\mathrm{area}\big[\,S_t^{\mathbf{x}} \cap C\,\big],$$

which is $1/(4\pi t)$ times the surface area of the intersection of the sphere of radius $t$ centered at $\mathbf{x}$ with the unit cube $C = \{\,0 \le x,y,z \le 1\,\}$.

(b) $\sqrt{2} < t < 3$ since the closest point to $(2,2,1)$ in $C$, namely $(1,1,1)$, is at a distance of $\sqrt{2}$, while the furthest, namely $(0,0,0)$, is at a distance of 3. The light signal starts out quiescent. Beginning at time $t = \sqrt{2}$, it gradually increases to a maximum value, which is closer to $\sqrt{2}$ than to 3, and then decreases, eventually dying out at $t = 3$.




(c) This is true for $t > 0$, but false for $t < 0$, since $u(-t,x,y,z) = -u(t,x,y,z)$ when $u(0,x,y,z) \equiv 0$.

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12.6.10. (a) $u(t,x,y) = \dfrac{1}{2\pi}\,\dfrac{\partial}{\partial t}\displaystyle\iint_{\|\boldsymbol{\xi}-\mathbf{x}\|\le t} \frac{\xi^3-\eta^3}{\sqrt{t^2-(\xi-x)^2-(\eta-y)^2}}\,d\xi\,d\eta.$

♢ 12.6.13. (a) $u(t,x) = \frac{1}{2}\,\delta(x-c\,t) + \frac{1}{2}\,\delta(x+c\,t)$; (b) $u(t,x) = \begin{cases} 1/(2\,c), & |\,x\,| < c\,t, \\ 0, & |\,x\,| > c\,t. \end{cases}$;

(c) Huygens' Principle is *not* valid in general for the one-dimensional wave equation, since, according to part (b), a concentrated initial velocity does not remain concentrated along the characteristics, but spreads out over all of space. Only concentrated initial displacements remain concentrated on characteristics.

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12.7.1. The atomic energy levels are multiplied by $Z^2$, so that formula (12.189) for the eigenvalues becomes $\lambda_n = -\dfrac{Z^2\alpha^4 M}{2\,\hbar^2}\,\dfrac{1}{n^2} = -\dfrac{Z^2\alpha^2}{2\,a}\,\dfrac{1}{n^2}$, $n = 1,2,3,\ldots$ .

♢ 12.7.3.

$$L_k^j(s) = \frac{s^{-j}e^s}{k!}\,\frac{d^k}{ds^k}\left[\,s^{j+k}e^{-s}\,\right] = \frac{s^{-j}e^s}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{d^{k-i}}{ds^{k-i}}\,s^{j+k}\,\frac{d^i}{ds^i}\,e^{-s}$$

$$= \frac{s^{-j}e^s}{k!}\sum_{i=0}^{k}\frac{k!}{i!\,(k-i)!}\,\frac{(j+k)!}{(j+i)!}\,s^{j+i}(-1)^i e^{-s} = \sum_{i=0}^{k}\frac{(-1)^i}{i!}\binom{j+k}{j+i}\,s^i.$$

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