Consider the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + y\dfrac{\partial u}{\partial x} = 0 & (t,x,y) \in R^{+} \times R \times R \\ u(0,x,y) = u_0(x,y) \end{cases}$$

Where $u_0(x,y)$ is a continuous function with $u_0(x,y) = 0$ for $|x|+|y| \geq 1$

(a) Solve the equation

(b) Prove that for fixed t_0 , $y_0 \in R$ $\lim\limits_{x\to\infty} u(t_0, x, y_0) = 0$

(a) $u = u(t,x,y)$

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

Characteristic lines

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = y, \frac{dy}{ds} = 0$$, t=s , $x = ys + x_0 = yt + x_0$

$$\frac{du}{ds} = 0$$ along the chracteritic lines

The general solution $u(t,x,y) = u_0(x - yt, y)$

(b)