§ Dispersion and Solitons

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§ 色散(dispersion)：

如果一個方程的相速度依賴於頻率或波數，就稱為色散方程。

例

KdV 方程 $u_t + uu_x + u_{xxx} = 0$ 線性化後色散關係為 $\omega = k^3, v_p = k^2$ (相速度依賴 k)是色散方程。($v_p = \dfrac{\omega}{k}$)

Schrodinger 方程 $iu_t + \Delta u = 0$ 平面波 $u(x,t) = e^{i(kx-\omega t)}$ 代入， $\omega = |k|^2$

因此 $v_p = |k|$ 不同頻率速度不同 $\rightarrow$ 色散

若相速度 $v_p = \dfrac{\omega}{k}$ 與 k 無關則為非色散方程。

也就是所有波長、傳播速度相同，波形保持不變。

例如 $u_{tt} = c^2 u_{xx}$ $u = e^{i(kx-\omega t)}$ $\omega^2 = c^2 k^2, v_p = \dfrac{\omega}{k} = c$ 是常數，是非色散方程。

§ 線性色散波

例 $u_t + u_{xxx} = 0$

$u(0,x) = f(x) = e^{-x^2}$ $-\infty < x < \infty$

要利用 Fourier 變換求解這個偏微分方程，我們可以將其從空間域 x 轉換到頻率域 ω 。

1. 對 PDE 作 Fourier 變換

   對 x 變量施加 Fourier 變換 $F\{u(t,x)\} = \hat{u}(t,\omega)$

   時間項 $F\{u_t\} = \dfrac{\partial \hat{u}}{\partial t}$ 空間項：根據微分性質 $F\{u_{xxx}\} = (i\omega)^3 \hat{u} = -i\omega^3 \hat{u}$

   代入原方程得到一個關於 t 的 ODE $\dfrac{\partial \hat{u}}{\partial t} - i\omega^3 \hat{u} = 0$

2. 求解頻域下的 ODE 得 $\hat{u}(t,\omega) = \hat{u}(0,\omega) e^{i\omega^3 t}$

3. 處理初始條件

   $u(0,x) = f(x) = e^{-x^2}$ ， $F\{e^{-ax^2}\} = \sqrt{\dfrac{\pi}{a}} e^{-\frac{\omega^2}{4t}}$

a=1 時， $\hat{u}(0,\omega)=\sqrt{\pi}e^{-\frac{\omega^2}{4}}$

代入頻域解中 $\hat{u}(t,\omega)=\sqrt{\pi}e^{-\frac{\omega^2}{4}}e^{i\omega^3 t}=\sqrt{\pi}\exp(i\omega^3 t-\frac{\omega^2}{4})$

4. 進行 Fourier 逆變換

$$u(t,x)=F^{-1}\{\hat{u}(t,\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(t,\omega)e^{i\omega x}d\omega$$

$$=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\frac{\omega^2}{4}+i(\omega x+\omega^3 t))d\omega$$

5. 最後 $u(t,x)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\frac{\omega^2}{4})\cos(\omega x+\omega^3 t)d\omega$

Ex

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8.5.1. Sketch a picture of the solution for the initial value problem in Example 8.13 at times $t=-.1,-.5$, and -1.

8.5.2. (a) Write down an integral formula for the solution to the dispersive wave equation (8.90) with initial data $u(0,x)=\begin{cases}1, & 0<x<1,\\ 0, & \text{otherwise.}\end{cases}$ (b) Use a computer package to plot your solution at several times and discuss what you observe.

8.5.3. (a) Write down an integral formula for the solution to the initial value problem

$$u_t+u_x+u_{xxx}=0, \qquad u(0,x)=f(x).$$

(b) Based on the results in Example 8.13, discuss the behavior of the solution to the initial value problem $u(0,x)=e^{-x^2}$ as t increases.

8.5.4. Find the (i) dispersion relation, (ii) phase velocity, and (iii) group velocity for the following partial differential equations. Which are dispersive? (a) $u_t+u_x+u_{xxx}=0$, (b) $u_t=u_{xxxxx}$, (c) $u_t+u_x-u_{xxt}=0$, (d) $u_{tt}=c^2u_{xx}$, (e) $u_{tt}=u_{xx}-u_{xxxx}$.

8.5.5. Find all linear evolution equations for which the group velocity equals the phase velocity. Justify your answer.

8.5.6. Show that the phase velocity is greater than the group velocity if and only if the phase velocity is a decreasing function of k for $k>0$ and an increasing function of k for $k<0$. How would you observe this in a physical system?

8.5.7. (*a*) *Conservation of Mass*: Prove that $T = u$ is a density associated with a conservation law of the dispersive wave equation (8.90). What is the corresponding flux? Under what conditions is total mass conserved? (*b*) *Conservation of Energy*: Establish the same result for the energy density $T = u^2$. (*c*) Is u^3 the density of a conservation law?

8.5.8. Prove that when $t = \pi p/q$, where p, q are integers, the solution (8.102) is constant on each interval $\pi j/q < x < \pi(j+1)/q$ for integers $j \in \mathbb{Z}$. *Hint*: Use Exercise 6.1.29(*d*). *Remark*: The proof that the solution is continuous and fractal at irrational times is considerably more difficult, [**90**].

8.5.9. (*a*) Find the complex Fourier series representing the fundamental solution $F(t, x; \xi)$ to the periodic initial-boundary value problem (8.100). (*b*) Prove that at time $t = 2\pi p/q$, where p, q are relatively prime integers, $F(t, x; \xi)$ is a linear combination of delta functions based at the points $\xi + 2\pi j/q$. *Hint*: Use Exercise 6.1.29(*c*). (*c*) Let $u(t, x)$ be any solution to (8.100). Prove that $u(2\pi p/q, x)$ is a linear combination of a finite number of translates, $f(x - x_j)$, of the initial data.