The Dirichlet conditionis a type of boundary condition commonly used in the study ofpartial differential equations (PDEs)。It specifies the values that a solution must take on the boundary of the domain。

For a PDE defined on a domain Ω with boundary $\partial\Omega$, theDirichlet boundary conditionis given by：

u(x)=f(x) for x∈ $\partial\Omega$

Examples

1. One-dimensional heat equation

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ ，0<x<L，t >0 with

Boundary condition $u(0,t) = T_1, u(0,L) = T_2$ for all t >0

Initial condition $u(x,0) = f(x)$ for $0 \le x \le L$

(1) Steady-state solution $u_s(x)$

(2) Transient solution (a time-dependent part)

…

$$u(x,t) = T_1 + \left(\frac{T_2 - T_1}{L}\right) x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha(n\pi/L)^2 t}$$

2. Temperature distribution in a square plate

   The steady-atate temperature u(x,y) in a square plate $\Omega = [0,\pi] \times [0,\pi]$ with the boundary condition：

   $u(0,y) = u(\pi,y) = u(x,0) = 0, u(0,\pi) = g(x)$

   …

   $$u(x,y) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$$

Condition	Mathematical Form	Physical Meaning
Dirichlet	$u = f$ on $\partial\Omega$	Fixed temperature/potential at boundary.
Neumann	$\frac{\partial u}{\partial n} = g$	Prescribed heat flux/flow.
Robin (Mixed)	$au + b\frac{\partial u}{\partial n} = h$	Convective cooling/heating.

3. Laplae equation in a circular domain (Polar coordinates)

   $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ for $0 \le r < R$ with the Dirichlet condition

   $u(R,\theta) = f(\theta)$

In the case $f(\theta)=\sin 3\theta$, $u(r,\theta)=(\frac{r}{R})^3\sin(3\theta)$

Comparison with Cartesian Coordinates:

Aspect	Cartesian (Rectangle)	Polar (Circle)
Domain	$[0, L_x] \times [0, L_y]$	$0 \le r \le R, \theta \in [0, 2\pi)$
Boundary	Edges $x = 0, x = L_x$, etc.	Circle $r = R$
Solution Form	$\sum \sin(\frac{m\pi x}{L_x}) \sin(\frac{n\pi y}{L_y})$	$\sum r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$
Singularities	None	Must exclude r^{-n} at $r = 0$

4. Laplace equation in a 3D sphere (spherical coordinates) a special case

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0,$$

A special case if $f(\theta,\phi)=f(\theta)=\cos\theta$ then

…

$$u(r,\theta)=\frac{r}{R}\cos\theta$$