§ Burger equation

TheBurger's equation is a fundamental partial differential equation (PDE) in fluid dynamics and **nonlinear wave theory**。It is a simplified model that combines nonlinearity and diffusion，and it is often used to study phenomena such as shock waves，turbulence，and traffic flow。

TheBurger's equationis given by： $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$

Where：

1. u(x,t)is the velocity field or the quantity of interest
2. t is time
3. xis the spatial coordinate
4. ν is the kinematic viscosity (a constant representing diffusion)。

Key Features:

1. Nonlinear Term $u\frac{\partial u}{\partial x}$：This term represents the nonlinear convection or advection, which causes wave steepening and can lead to the formation of shocks.
2. Diffusion Term $\nu\frac{\partial^2 u}{\partial x^2}$：This term represents viscous(黏稠的) diffusion，which smooths out sharp gradients and prevents the formation of infinite discontinuities。

The Burger's equation is a prototypical(原型的) example of a PDE that balances nonlinearity and diffusion，making it a key equation in the study of nonlinear dynamics and applied mathematics。

Suppose u is a classical solution to the following initial-boundary value problem for a viscous Burger equation

$$\begin{cases} u_t+uu_x=u_{xx},(x,t)\in(0,1)\times(0,\infty) \\ u(0,t)=u(1,t)=0,t>0 \\ u(x,0)=\text{g(x)},\text{x}\in[0,1] \end{cases}$$

Here ，g(x) is a smooth function such that g(0)=g(1)=0

Show that

(a) $\int_0^1 u^2(x,t)dx$ is decreasing (b) $\int_0^1 u^2(x,t)dx$ tends to 0 exponentially。