The KdV equation：

§ 01 a wave solution

$u_t + 6uu_x + u_{xxx} = 0 ...(1)$ ，$-\infty < x < \infty, 0 \le t < \infty$ with initial condition $u(x,0) = f(x)$

Is used to describe the evolution of shallow water wave。

A traveling wave solution which has permanent form occurs to a balance of its dispersive (相散)term $u_{xxx}$，and its nonlinear term $6uu_x$。

Let $\xi = x - ct$

Let $u(x,t) = f(x-ct)$ ，$u_t = \frac{df}{d\xi}\frac{d\xi}{dt} = -c\frac{df}{d\xi}$ then $-cf' + 6ff' + f''' = 0 ...(2)$

(2) 積分一次得 $-cf + 3f^2 + f'' = A$ 將 $f'$ 視為積分因子(即兩邊同乘以 $f'$)

$f'f'' = Af' + cff' - 3f^2 f'$

$[\frac{1}{2}(f')^2]' = (Af)' + (\frac{c}{2}f^2)' - (f^3)'$ 再積分得

$(f')^2 = 2Af + cf^2 - 2f^3 + B$

考慮邊界值 $f, f', f'' \to 0$ as $x \to \infty$ then A=B=0

$(f')^2 = cf^2 - 2f^3 = f^2(c - 2f)$

$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \pm \int d\xi$ ，let $f = \frac{c}{2}\operatorname{sech}^2\theta$ then $c - 2f = ... = c\tanh^2\theta$

其中

$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}, \cosh^2 x - \sinh^2 x = 1$ ，$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$

$f' = \frac{df}{d\xi}, \xi = x - ct$ ，$df = c\operatorname{sech}\theta(-\tanh\theta\operatorname{sech}\theta)d\theta$

$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = ... = \frac{-2\theta}{\sqrt{c}} = \pm(\xi + k) = \pm(\text{x-ct+x}_0)$ ，$\theta = \frac{\sqrt{c}}{2}(x - ct + x_0)$

$u(x,t) = f(x-ct) = \frac{c}{2}\operatorname{sech}^2[\frac{\sqrt{c}(x-ct+x_0)}{2}]$

§ 02 the time-independent *Schrödinger* equation

Miura transform $u = v^2 + v_x ...(3)$ ，代入(1)得

$(2v + \frac{\partial}{\partial x})(v_t - 6v^2 v_x + v_{xxx}) = 0$

Miura transformation 看作是 v 的 Riccatti equation

Let $v=\frac{\psi'}{\psi}$ ，then (3)變成 $\psi_{xx}-u\psi=0$ here $\psi'=\frac{d\psi}{dx}$

可把此式改寫成

$\psi_{xx}+(\lambda-u)\psi=0$ $Schr\ddot{o}dinger$ equation

Where u(x,t) plays the role of a potential and $\lambda$ is an eigenvalue of $\psi(x,t)$

§ 03 Inverse scattering transform(逆散射變換)理論 for KdV

1967 年 Gardner Greene Martin Kruskal Robert Miura

$u_t+6uu_x+u_{xxx}=0$ 的 Fourier transform 為

$$\hat{u}(k,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{-ikx}dx$$

1974 年把此方法推廣 稱 AKNS scheme 首先解了 sine-Gordon 方程。

If one is given a linear pde with some initial condition, then the solution of the linear pde can be determine using the following steps:

- take the Fourier transform of the linear pde which results in a linear ordinary differential equation (ode) in the Fourier space
- take the Fourier transform of the initial condition (usually not too difficult)
- solve the resulting ode with its initial condition in Fourier space
- transform back to physical space and obtain the solution in the original variables.

Example

$u_t=u_{xx}, u(x,0)=f(x), -\infty<x<\infty$

Where u=u(x,t) and f(x) has a Fourier transform。

Define Fourier transform $F(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$

and inverse Fourier transform $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{ikx}dk$

$\int_{-\infty}^{\infty}u_t e^{-ikx}dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(ue^{-ikx})dx=\frac{\partial}{\partial t}\int_{-\infty}^{\infty}ue^{-ikx}dx=\frac{\partial U}{\partial t}$ ，where U is the Fourier transform of u(x,t)

$\int_{-\infty}^{\infty}u_{xx}e^{-ikx}dx=\cdots=-k^2U$ (其中作兩次 integrate by parts with respect to x)

原式變成 $\frac{\partial U}{\partial t}=-k^2U, U(k,0)=F(k), -\infty<k<\infty$

$$U(k,t)=U(k,0)e^{-k^2t}$$

取 inverse Fourier transform⋯

§ Lax Method in Hilbert space

§ Galilean invariant

Galilean transformation：

A uniform motion $(x,t)\to(x+tv,t)$

A translation $(x,t)\to(x+a,t+s)$

A rotation $(x,t)\to(Rx,t)$

The transformation which describes Galilean invariance is given by：

$x=x'+6\lambda t', t=t', u=u'-\lambda, -\infty<\lambda<\infty$ ..(*)

⋯

$\tilde{u}_T+6\tilde{u}\tilde{u}_X+\tilde{u}_{XXX}=0$

The KdV equation is invariant under the transformation given by (*)。

§ The conservation law

Consider the equation $\frac{\partial T}{\partial t}+\frac{\partial X}{\partial x}=0$ where $T=T(x,t,u,u_x,u_{xx},\ldots)$ and

$X=X(x,t,u,u_x,u_{xx},\ldots)$ If $X\to 0$ as $|x|\to\infty$

Then $\frac{d}{dt}(\int_{-\infty}^{\infty}Tdx)=0$，implies that $\int_{-\infty}^{\infty}Tdx$ =constant。

$u_t+6uu_x+u_{xxx}=0$

$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(3u^2+u_{xx})=0$ then $\int_{-\infty}^{\infty}u(x,t)dx$ =constant，where we have taken

$u,u_x,u_{xx}\to 0$ as $|x|\to\infty$