§ Phase flows

Definition. A *one-parameter diffeomorphism group* is a one-parameter transformation group whose elements are diffeomorphisms satisfying the additional condition that $g^t x$ depends smoothly on both of the arguments t and x.

Example 1. $M = \boldsymbol{R}$, g^t is multiplication by e^{kt}.

Example 2. $M = \boldsymbol{R}^2$, g^t is rotation about 0 by the angle t.

Definition. A *one-parameter group of linear transformations* is a one-parameter diffeomorphism group whose elements are linear transformations.

Example. On the plane with coordinates (x, y) consider the transformation $g^t(x, y) = (e^{\alpha t}x, e^{\beta t}y)$.

For a differential equation $\dot{x} = v(x)$，the phase flow is a one-parameter diffeomorphism group for which v is the phase velocity vector field $v(x) = \frac{d}{dt}\Big|_{t=0}(g^t x)$ 。

1. 分別求(1) $\dot{x} = \sin x, 0 < x < \pi$ (2) $\begin{cases} \dot{x} = y \\ \dot{y} = 1 \end{cases}$ 的 phase flows

   (1) φ_t is the flow of vector field v

   Then $\dot{\varphi} = \sin\varphi$，$\dfrac{d\varphi}{\sin\varphi} = dt$

   兩邊積分 $\displaystyle\int \frac{d\varphi}{\sin\varphi} = -\ln\left|\cot\frac{\varphi}{2}\right| = t + c$，with $\varphi_0(x) = x$ 解出

   $\varphi_t(x) = 2\,\mathrm{arccot}(e^{-t}\cot\frac{x}{2})$

   Note that $\int \csc s dx = -\ln\left|\csc x + \cot x\right| + c$

   $$\csc x + \cot x = \frac{1+\cos x}{\sin x} = \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\frac{x}{2}$$

(2) 即 $\begin{cases} \dot{\varphi}_t^{\,1} = \varphi_t^2 \\ \dot{\varphi}_t^{\,2} = 1 \end{cases}$

由(2) $\varphi_t^2 = y + t$ with $\varphi_0^2(y) = y$

代入(1) $\dot{\varphi}^1 = y + t$ ， $\varphi^1 = ty + \frac{1}{2}t^2 + x$

Thus the phase flow of the system is $g_{(x,y)}^t = (x + ty + \frac{1}{2}t^2, y + t)$

習作

(1) Find the phase flow of $\dot{x} = kx$ ，k>0 $\varphi_t = xe^{kt}$

(2) Find the phase flow of $\dot{x} = x - 1$ $\varphi_t(x) = (x-1)e^t + 1$

(3) Find the phase flow of $\begin{cases} \dot{x} = \sin y \\ \dot{y} = 0 \end{cases}$ $\varphi^t(x, y) = (x + t\sin y, y)$

↰

(1) $\dot{\varphi} = k\varphi$ with $\varphi_0 = id$ i.e. $\varphi_0(x) = x$

Then $\varphi_t(x) = xe^{kt}$

(2) $\dot{\varphi}_t(x) = \varphi_t(x) - 1, \varphi_0(x) = x$

$\varphi_t(x) = ke^t + 1$ ，k=x-1 ，$\because \varphi_0(x) = x$

$\therefore \varphi_t(x) = (x-1)e^t + 1$

(3) $\dot{\varphi}_2^t = 0, \varphi_2^t = y$

$\dot{\varphi}_1 = \sin y$ ， $\varphi_1^t = t\sin y + x$

所以 $g_{(x,y)}^t = (x + t\sin y, y)$

Q：是否每一個 smooth vector field 是一個 flow 的 phase velocity vector field？

The phase velocity vector field $v(x)=\frac{d}{dt}\Big|_{t=0}(g^t x)$

反例 $v(x)=x^2$

對一個特定的 x_0，解出 $x=\frac{x_0}{1-x_0 t}$ 所以 $g^t x=\frac{x}{1-tx}$ 容易驗證 $g^{t+s}=g^t(g^s)$ 但是 g^t is not a diffeomorphism of the line for any value of t except t=0

所以 v(x)沒有 phase flow。

Q：$v(x)=x^2$ rectify(拉直)the direction field in a neighborhood of the origin。(Rectification Theorem)

註 1：$\dot{x}=x^2$ 與 $v(x)=x^2$ 是同一件事

$\varphi_t(x)$ 是 $v(x)=x^2$，則(1) $\dot{\varphi}=\varphi^2$ 且(2) $\varphi_0(x)=x$ 得到 $\varphi_t(x)=\frac{x}{1-tx}$

而 $\dot{x}=x^2$ 得 $x=\frac{-1}{t+c}$ 前者給了 initial condition

For a fixed point x_0，$\varphi(t)=g^t x_0$，where $\varphi: R\to M$

$\varphi(t)$ is a solution of the equation $\dot{x}=v(x)$ with the initial condition $\varphi(0)=x_0$

一個微分方程的解是局部的，而所謂流線(flow)是定義在整個 R，換句話說是定義於所有時間 t。

註 2：A smooth vector field on a compact manifold always defines a phase flow。(So the vector field is complete。)

Q：Does the equation $\dot{x}=e^x\sin x$ define a phase flow on the line？ Yes

2. $v=x\frac{\partial}{\partial x}$，$y=g(x)=e^x$，find the image of v under the action of g

$\dot{x}=v(t,x)=x$ 表示 v 的 flow φ_t 滿足 $\dot{\varphi}_t=\varphi_t$ 且 $\varphi_0(x)=x$

則 $\varphi_t=xe^t$，$g(\varphi(t))=e^{xe^t}$

$$\frac{d}{dt}\Big|_{t=0} g(\varphi(t))=(xe^t)e^{xe^t}\Big|_{t=0}=xe^x=y\ln y \quad (\because y=e^x)$$

Thus the image of v is $y\ln y\frac{\partial}{\partial y}$