§ The inverse Laplace transform

$$F(s)=\int_0^\infty e^{-st} f(t)dt \Leftrightarrow f(t)=L^{-1}\{F(s)\}$$

Convolution integral(褶積 捲積 疊積)

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau \quad \text{定理} \quad L\{f*g)\}=L\{f\}L\{g\}$$

Both L and L^{-1} are linear operators：

$$L^{-1}\{af(t)+bg(t)\}=aL^{-1}\{f(t)\}+bL^{-1}\{g(t)\}$$

f(t)	1	t	t^n	$e^{\alpha t}$	$\sin\lambda t$	$\cos\lambda t$	$\sinh\lambda t$	$\cosh\lambda t$	$t\sin\lambda t$
F(s)	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-\alpha}$	$\frac{\lambda}{s^2+\lambda^2}$	$\frac{s}{s^2+\lambda^2}$	$\frac{\lambda}{s^2-\lambda^2}$	$\frac{s}{s^2-\lambda^2}$	$\frac{2\lambda s}{s^2+\lambda^2}$

f(t)	te^{-at}	$f'(t)$	tf(t)
F(s)	$\frac{1}{(s+a)^2}$	sF(s)-f(0)	$-F'(s)$

$$L\{y'(t)\}=sF(s)-y(0)$$
$$L\{y''(t)\}=s^2F(s)-sy(0)-y'(0)$$

Examples

1. 解 $y''+2y'+2y=\sin\alpha t, y(0)=0, y'(0)=0$

   $\{s^2F(s)-sy(0)-y'(0)\}+2\{sF(s)(0)\}+2F(s)=L\{\sin\alpha t\}$

   $(s^2+2s+2)F(s)=L\{\sin\alpha t\}$

   $$F(s)=\frac{1}{(s+1)^2+1}L\{\sin\alpha t\}=L\{e^{-t}\sin t\}L\{\sin\alpha t\}=L\{(e^{-t}\sin t)*\sin\alpha t)\}$$

   $$y(t)=(\sin\alpha t)*(e^{-t}\sin t)=\int_0^t \sin\alpha(t-\tau)e^{-\tau}\sin\tau d\tau$$

2. 解 $y''-3y'+4y=0, y(0)=1, y'(0)=5$

   $L\{y''\}=s^2F(s)-sy(0)-y'(0)=s^2F(s)-s-5$

   $L\{y'\}=sF(s)-y(0)=sF(s)-1$

   …

   $$F(s)=\frac{s+2}{s^2-3s+4}=\frac{(s-\frac{3}{2})+\frac{7}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}}=L\{e^{\frac{3}{2}t}\cos\frac{\sqrt{7}}{2}t\}+L\{e^{\frac{3t}{2}}\cdot\sqrt{7}\cdot\sin\frac{\sqrt{7}t}{2}\}$$

   $$\therefore y(t)=e^{\frac{3t}{2}}(\cos\frac{\sqrt{7}t}{2}+\sqrt{7}\sin\frac{\sqrt{7}t}{2})$$

3. 解 $\begin{cases} x' = 2x - 3y \\ y' = -2x + y \end{cases}, x(0) = 8, y(0) = 3$ $\quad \begin{cases} x(t) = 5e^{-t} + 3e^{4t} \\ y(t) = 5e^{-t} - 2e^{4t} \end{cases}$

Let $L\{x\} = X(s), L\{y\} = Y(s)$ then

$$\begin{cases} sX(s) - 8 = 2X(s) + 3Y(s) \\ sY(s) = -2X(s) + Y(s) \end{cases}$$

$$\begin{cases} X(s) = \dfrac{8s-17}{s^2-3s-4} = \dfrac{5}{s+1} + \dfrac{3}{s-4} \\ Y(s) = \dfrac{3s-22}{s^2-3s-4} = \dfrac{5}{s+1} - \dfrac{2}{s-4} \end{cases} \qquad L\{e^{\alpha t}\} = \frac{1}{s-\alpha}, s > \alpha$$

$$\begin{cases} x(t) = 5e^{-t} + 3e^{4t} \\ y(t) = 5e^{-t} - 2e^{4t} \end{cases}$$

4.

Examples

1. If $F(s) = \dfrac{1}{s+4} + \dfrac{s}{s^2-9}$, $L^{-1}\{F(s)\} =$

$f(t) = e^{-4t} + \cosh 3t$

2. $L^{-1}\{\dfrac{1}{(s+1)^2}\} =$ $\quad te^{-t}$

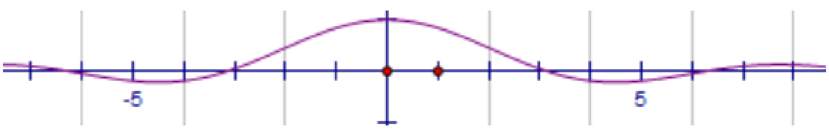
3. (a) $L^{-1}\{\dfrac{1}{s^2-1}\} =$ (b) $L^{-1}\{\dfrac{s}{s^2+9}\} =$

(a) $L^{-1}\{\dfrac{1}{s^2-1}\} = \sinh t$ (b) $L^{-1}\{\dfrac{s}{s^2+9}\} = \cos 3t$

4. $L^{-1}\{\dfrac{3s+8}{s^2+2s+5}\}$ $\quad e^{-t}(3\cos 2t + \dfrac{5}{2}\sin 2t)$

5. $L^{-1}\{\dfrac{3s+2}{s^2-3s+2}\} =$ $\quad -5e^{t} + 8e^{2t}$

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ $\quad y = \dfrac{\sin x}{x}$的圖形：



$\int_0^\infty \frac{\sin tx}{x} dx =$

假設 $\int_0^\infty \frac{\sin tx}{x} dx = f(t)$ 則

$\mathrm{L}(s, f) = \int_0^\infty e^{-st} (\int_0^\infty \frac{\sin tx}{x} dx) dt = \int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dx dt = \int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dt dx$

$= \int_0^\infty (\int_0^\infty e^{-st} \sin tx dt) \frac{dx}{x} = \int_0^\infty (L(s, \sin xt) \frac{dx}{x} = \int_0^\infty \frac{x}{s^2 + x^2} \frac{dx}{x}$

$= \int_0^\infty \frac{1}{s^2 + x^2} dx = \frac{1}{s} (\arctan \frac{x}{s}) \Big|_0^\infty = \frac{1}{s} \times \frac{\pi}{2} = L(s, \frac{\pi}{2})$

所以 $f(t) = \frac{\pi}{2}$

其中 $L(s,1) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$ ，所以 $L(s, \frac{\pi}{2}) = \frac{\pi}{2} \times \frac{1}{s}$

所以 $\int_0^\infty \frac{\sin x}{x} dx = f(1) = \frac{\pi}{2}$

另解

$L\{s, \sin \lambda t\} = \frac{\lambda}{s^2 + \lambda^2}$

若 $\lim_{t \to 0} \frac{f(t)}{t}$ 存在則 $\int_s^\infty F(u) du = L(s, \frac{f(t)}{t})$

$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{\sin t}{t} (\lim_{s \to 0} e^{-st}) dt = \lim_{s \to 0} \int_0^\infty e^{-st} \frac{\sin t}{t} dt$

$= \lim_{s \to 0} L\{\frac{\sin t}{t}\} = \lim_{s \to 0} \int_s^\infty F(u) du$ ，其中 $F(u) = L\{\sin u\} = \frac{1}{u^2 + 1}$

$= \lim_{s \to 0} \int_s^\infty \frac{1}{1 + u^2} du = \lim_{s \to 0} \tan^{-1} \Big|_s^\infty = \frac{\pi}{2}$