§ Laplace transform $L\{f(t)\} = \mathrm{F}(s) = \int_0^{\infty} e^{-st} f(t)dt$

properties：

1. 線性 $L\{af + bg\} = aL\{f\} + bL\{g\})$
2. 微分 L{f'(t)}=sF(s)-f(0)

   $L\{y'(t)\}$=sF(s)-y(0) ，$L\{y''(t)\} = s^2F(s) - sy(0) - y'(0)$
3. 積分 $L\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$
4. 位移 $L\{e^{at} f(t)\} = F(s-a)$
5. $L\{tf(t)\} = -\frac{d}{ds}F(s)$ 即 $\frac{d}{ds}\int_0^{\infty} e^{-st} f(t)dt = -L\{tf(t)\}$
6. 若 $\lim_{t\to 0}\frac{f(t)}{t}$ 存在，則 $L(s, \frac{f(t)}{t}) = \int_s^{\infty} L(s, f)ds$

$\cosh t = \frac{e^t + e^{-t}}{2}, \sinh t = \frac{e^t - e^{-t}}{2}$

$L(s, \cos\lambda t) = \frac{s}{s^2+\lambda^2}$ $L(s, \sin\lambda t) = \frac{\lambda}{s^2+\lambda^2}$

$L(s, \cosh\lambda t) = \frac{1}{s^2-\lambda^2}$ $L(s, \sinh\lambda t) = \frac{\lambda}{s^2-\lambda^2}$

Examples

1. $L(s,1) = \int_0^{\infty} e^{-st}dt = \frac{1}{s}$
2. $L\{t\} = \int_0^{\infty} e^{-st}tdt = \frac{1}{s^2}$
3. $L\{t^n\} = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}}, n > -1 \\ \frac{n!}{s^{n+1}}, n \in N \end{cases}$
4. $L\{e^{\alpha t}\} = \frac{1}{s-\alpha}$ for $\alpha < s$
5. $L\{\cos\lambda t\} = \frac{s}{s^2+\lambda^2}$ [DE8001Laplace-1]
6. Prove that $L\{tf(t)\} = -\frac{d}{ds}F(s)$

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt = -\int_0^\infty e^{-st} tf(t)dt = -L(s, tf(t))$$

7. Prove that $\int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2}, t > 0$

Let $\int_0^\infty \frac{\sin tx}{x} dx = f(t)$ then $L\{f\} = \int_0^\infty e^{-st} (\int_0^\infty \frac{\sin tx}{x} dx)dt = \int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dxdt$

$$= \int_0^\infty \int_0^\infty e^{-st} \frac{sintx}{x} dtdx = \int_0^\infty (\int_0^\infty e^{-st} \sin xtdt) \frac{dx}{x} = \int_0^\infty \mathrm{L(s,sinxt)} \frac{dx}{x}$$

$$= \int_0^\infty \frac{x}{s^2 + x^2} \frac{dx}{x} = \frac{1}{s}(\tan^{-1} \frac{x}{s})\Big|_0^\infty = \frac{1}{s} \times \frac{\pi}{2} = L(s, \frac{\pi}{2}) \qquad L(s\ ,1) = \frac{1}{s}$$

$$\therefore f(t) = \frac{\pi}{2}$$

8. $L\{e^{4t}t^6\} =$

$L\{e^{4t}t^6\} = F(s-4) = \int_0^\infty e^{-(s-4)t} t^6 dt = \int_0^\infty t^6 e^{-at} dt$ ，a=s-4，s>4

Let x=at $\qquad = \int_0^\infty (\frac{x}{a})^6 e^{-x} (\frac{1}{a})dx = \frac{1}{a^7} \int_0^\infty x^6 e^{-x} dx = a^{-7} \times \Gamma(7) = \frac{6!}{(s-4)^7}$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

9. $\mathrm{L}\{(t+1)^2 e^t\} =$ $\qquad L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$, $s > a$

$$L\{(t+1)^2 e^t\} = L\{t^2 e^t\} + 2L\{te^t\} + \mathrm{L}\{e^t\} = \frac{\Gamma(3)}{(s-1)^3} + \frac{2\Gamma(2)}{(s-1)^2} + \frac{1}{s-1}$$

Exercises

1. $L\{t \sin \omega t\} = \frac{2s}{(s^2 + \omega^2)^2}$
2. $L\{\frac{\sin \omega t}{t}\} = \tan^{-1} \frac{1}{s}$
3. $L\{e^{at} \sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}$