§ Linear Systems $x' = Ax$

Q：以下求e^{tA}

1. $A=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ $\qquad \begin{pmatrix}e^{t}&0\\0&e^{2t}\end{pmatrix}$

2. $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\qquad \begin{pmatrix}1&t\\0&1\end{pmatrix}$

3. $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $\qquad \begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}$

4. $A=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ $\qquad \begin{pmatrix}1&t&\frac{1}{2}t^{2}\\0&1&t\\0&0&1\end{pmatrix}$

5. $A=\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}$ $\qquad \begin{pmatrix}e^{\lambda t}&te^{\lambda t}\\0&e^{\lambda t}\end{pmatrix}$

6. $A=\begin{pmatrix}3&-2\\2&-2\end{pmatrix}$ $\qquad \begin{pmatrix}-\frac{1}{3}e^{-t}+\frac{4}{3}e^{2t}&\frac{2}{3}e^{-t}-\frac{2}{3}e^{2t}\\-\frac{2}{3}e^{-t}+\frac{2}{3}e^{2t}&\frac{4}{3}e^{-t}-\frac{1}{3}e^{2t}\end{pmatrix}$

例 1 $x'=\begin{pmatrix}3&-2\\2&-2\end{pmatrix}x, A=\begin{pmatrix}3&-2\\2&-2\end{pmatrix}$ ，$\lambda=-1,2$

$\lambda_1=-1,\xi=\begin{pmatrix}1\\2\end{pmatrix},\lambda_2=2,\eta=\begin{pmatrix}2\\1\end{pmatrix}$

$x^{1}=\begin{pmatrix}1\\2\end{pmatrix}e^{-t},x^{2}=\begin{pmatrix}2\\1\end{pmatrix}e^{2t}$ general solution is $x(t)=c_1\begin{pmatrix}2\\1\end{pmatrix}e^{-t}+c_2\begin{pmatrix}1\\2\end{pmatrix}e^{2t}$

$A=\begin{pmatrix}3&-2\\2&-2\end{pmatrix},T=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ ，則$T^{-1}AT=\begin{pmatrix}-1&0\\0&2\end{pmatrix}=D$ ，$T^{-1}=\frac{1}{3}\begin{pmatrix}1&-2\\-2&1\end{pmatrix}$

$$T^{-1}e^{tA}T=e^{T^{-1}tAT}=e^{tD}=\begin{pmatrix}e^{-t}&0\\0&e^{2t}\end{pmatrix}$$

$$e^{tA}=Te^{tD}T^{-1}=-\frac{1}{3}\begin{pmatrix}e^{-t}-4e^{2t}&-2e^{-t}+2e^{2t}\\2e^{-t}-2e^{2t}&-4e^{-t}+e^{2t}\end{pmatrix}$$

$x' = Ax, x(0) = x^0$ ，則 $x = x^0 \exp(tA)$ 取 $x^0 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ 則 $x = x^0 e^{tA} = \ldots = \begin{pmatrix} e^{-t} + 2e^{2t} \\ 2e^{-t} + e^{2t} \end{pmatrix}$

然後把 e^{tA} 稱為 one-parameter group

Fundamental matrix $\psi(t)$

general solution is $x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

取 $x^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix}$ ，$c_1 = -\frac{1}{3}, c_2 = \frac{2}{3}$ ，$x^{(1)}(t) = -\frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \frac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

$x^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix}$ ，$c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}$ ，$x^{(2)}(t) = \frac{2}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} - \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

則 $\psi(t) = e^{tA} = \begin{pmatrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} \end{pmatrix}$

$x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x, A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ ，$\psi(t) = e^{tA} = \begin{pmatrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} \end{pmatrix}$

Abel 定理

$x^{(1)}, x^{(2)}, \ldots$ 是 $x' = Ax$ 的 fundamental set of solutions 則稱

$\psi(t) = (x_j^{(i)})$ 為 $x' = Ax$ 的 fundamental matrix

$W(x^{(1)}, x^{(2)}, \ldots) = \det \psi(t)$ ，則 $\frac{dW}{dt} = (trA)W, \det \psi' = (\det A)W$

$W(x^1, x^2, \ldots, x^n)(t_0) \neq 0 \Leftrightarrow W(x^1, x^2, \ldots, x^n)(t) \neq 0$ for $\forall t \in R$

$\Leftrightarrow x^1, x^2, \ldots, x^n$ linearly independent

取 $\psi(t) = \begin{pmatrix} e^{-t} & 2e^{-t} \\ 2e^{2t} & e^{2t} \end{pmatrix}$ ，則 $W = \det \psi = -3e^{t}$ ，tr(A)=1

ψ(t) 是 $x' = Ax$ 的解，$\psi' = A\psi$ 所以 $\det \psi' = (\det A)(\det \psi) = (\det A)W$

$\frac{dW}{dt} = (trA)W$ 意義如何？

例 2 $\begin{cases} y' = 3y - z \\ z' = y + z \end{cases}$ ， $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ ， $\lambda = 2, 2$

$\lambda = 2, \xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

考慮另一解為 $x = \xi t e^{2t}$ ，則 $x' = \xi(e^{2t} + 2te^{2t}) = A(\xi t e^{2t}) = 2\xi t e^{2t}$ ， $\xi = 0$ 不合

所以假設 $x = \xi t e^{2t} + \eta e^{2t}$ 則

$x' = \xi(e^{2t} + 2te^{2t}) + \eta(2e^{2t}) = A(\xi t e^{2t} + \eta e^{2t}) = \xi(2te^{2t}) + A\eta e^{2t}$ （ $A\xi = 2\xi$ ）

$\begin{cases} (A - 2I)\eta = \xi \ldots (1) \\ (A - 2I)\xi = 0 \ldots (2) \end{cases}$ (2)是顯然的

由(1) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ， $\eta_1 - \eta_2 = 1$ ，let $\eta_2 = k$ 則 $\eta = \begin{pmatrix} k+1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ ，其中 $k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ 吸收到 $x^{(1)}(t)$ 中

$\therefore x^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$

$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$

$y(t) = (c_1 + c_2 t)e^{2t} + c_2 e^{2t}$

$z(t) = (c_1 + c_2 t)e^{2t}$

習作

1. 解 $\begin{cases} \dfrac{dx}{dt} = x + y + z \\ \dfrac{dy}{dt} = 2x + y - z \\ \dfrac{dz}{dt} = -3x + 2y + 4z \end{cases}$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$ ， $\lambda = 2, 2, 2$

$x^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$ ， $x^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$ ， $x^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} e^{2t}$

General solution is $x = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + c_3 x^{(3)}(t)$

2.