§ A spring-mass system

$my''+cy'+ky=F$ ...the equation of motion

Hooke's law $F_s = k\Delta L = k(\Delta l - y)$

Damping force $F_d = -cy'$

Gravity =–mg $g = 9.8m/s^2 = 32\,ft/s^2$

Extenal force F

y is the displacement of the object from its equilibrium position。

Newton's second law of motion：

$my'' = -mg + F_d + F_s + F$

Example 6.1.1 An object stretches a spring 6 inches in equilibrium.

(a) Set up the equation of motion and find its general solution.

(b) Find the displacement of the object for $t > 0$ if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

$\Delta l = 6inchs = \frac{1}{2}ft$

$my''+ky=0 \Rightarrow y''+\omega_0{}^2 y=0$ then $\omega_0 = \sqrt{\frac{k}{m}}$

$mg = k\Delta l \quad \frac{k}{l} = \frac{g}{\Delta l} = \frac{32}{1/2} = 64$

$y''+64y=0$

$y = c_1\cos 8x + c_2 \sin 8x$

(b) 18 inch=$\frac{3}{2}$ ft ， $y(0)=\frac{3}{2}, y'(0)=-3$

...

$y = R\cos(\omega_0 t - \phi)$ ，period $T = \frac{2\pi}{\omega_0}$ ...simple harmonic motion

Example 6.1.3 The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

$g = 980cm/s^2$ ， $\Delta l = 2cm$

$my''+ky=0 \to y''+\omega_0{}^2 y=0$

By Hooke's law $F_s = k\Delta L \Rightarrow mg = k\Delta l \quad \omega_0{}^2 = \frac{k}{m} = \frac{g}{\Delta l} = 490$

$y''+490y=0, y(0)=-1, y'(0)=14$

...

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}}\sin 7\sqrt{10}t$$

Frequency= $7\sqrt{10} rad/s$ ，period $T = \dfrac{2\pi}{7\sqrt{10}}$ sec

the phase angle ϕ is defined by $\cos\phi = \dfrac{c_1}{R}$ $\sin\phi = \dfrac{c_2}{R}$

$my'' + ky = F_0 \cos \omega t$ with $\omega_0 = \sqrt{\dfrac{k}{m}}$

Example 6.1.4 Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t, \quad y(0) = 0, \quad y'(0) = 0,$$

given that $\omega \neq \omega_0$.

$y = R(t)\sin\dfrac{(\omega_0 + \omega)t}{2}, R(t) = \ldots$

§ 6.2

free vibrations with damping

$my'' + cy' + ky = 0$

Example 6.2.1 Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

(a) Write the equation of motion of the object and determine the value of c for which the motion is critically damped.

(b) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.

(c) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

(a) 若 1 磅力（lbF）的力作用在某物體上，此物體獲得 1 英尺/秒 ^{2}（1 ft/s^2）的加速度，則此物體的質量是 1 斯勒格。

$mg = k\Delta l = 64lb$ where $g = 32lb/s^2$ $\Delta l = 6inches = \dfrac{1}{2}ft$

$\therefore$ m=2 slugs $k = 128lb/ft$

$2y'' + cy' + 128y = 0$

The damping is critical if c=32(lb-sec/ft)

(b) $y'' + 16y' + 64y = 0$

$y = e^{-8t}(c_1 + c_2 t)$

(1) $y(0)=1, y'(0)=20 \Rightarrow c_1=-8, c_2=28$

(2) $y(0)=1, y'(0)=-20 \Rightarrow c_1=-8, c_2=-12$

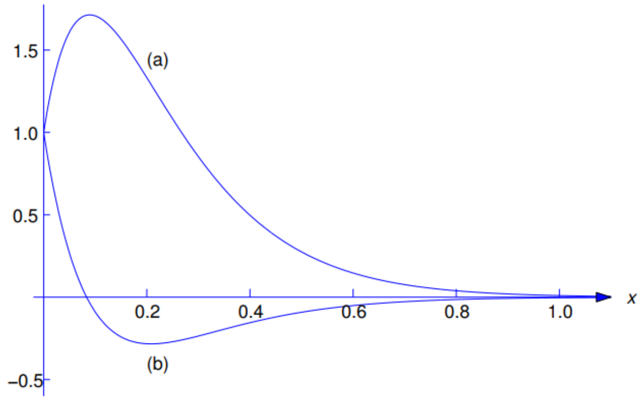

Figure 6.2.2 **(a)** $y=e^{-8t}(1+28t)$ **(b)** $y=e^{-8t}(1-12t)$

forced vibrations with damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t)=F_0\cos\omega t$, so that the equation of motion is

$$my''+cy'+ky=F_0\cos\omega t. \qquad (6.2.11)$$

Theorem 6.2.1 *Suppose we consider the amplitude R of the steady state component of the solution of*

$$my''+cy'+ky=F_0\cos\omega t$$

as a function of ω.

(a) *If $c \geq \sqrt{2mk}$, the maximum amplitude is $R_{\max}=F_0/k$ and it's attained when $\omega=\omega_{\max}=0$.*

(b) *If $c<\sqrt{2mk}$, the maximum amplitude is*

$$R_{\max}=\frac{2mF_0}{c\sqrt{4mk-c^2}}, \qquad (6.2.20)$$

and it's attained when

$$\omega=\omega_{\max}=\sqrt{\frac{k}{m}\left(1-\frac{c^2}{2km}\right)}. \qquad (6.2.21)$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of c, for $c \geq 0$, since (6.2.20) and (6.2.21) reduce to $R_{\max}=F_0/k$ and $\omega_{\max}=0$ if $c=\sqrt{2km}$.

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8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t>0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

虎克定律 $F_s=mg=k\Delta l \quad \therefore k=\frac{mg}{\Delta l}=\frac{20\times 980}{5}=3920$

$20y''+400y'+3290y=0 \Rightarrow y''+20y'+196y=0$ with $y(0)=9, y'(0)=0$