§ variation of parameters

$p(x)y''+q(x)y'+r(x)y=f(x)$ ...(*)

If we know a fundamental set $\{y_1, y_2\}$ of the complementary equation

$p(x)y''+q(x)y'+r(x)y=0$ ...(**)

Having found a particular solution y_p of (*)，we can write the general solution of

(*) as $y=y_p+c_1y_1+c_2y_2$

例 $x^2y''-2xy'+2y=x^{\frac{9}{2}}\cdots(*)$

已知 $y_1=x, y_2=x^2$ 是補充方程 $x^2y''-2xy'+2y=0$ 的解

Let $y_p=u_1x+u_2x^2$ then

$y_p'=u_1+2xu_2+u_1'x+u_2'x^2$， let $u_1'x+u_2'x^2=0$ then $y_p'=u_1+2xu_2$

$y_p''=u_1'+2u_2+2xu_2'$

代入(*) 化簡 得 $u_1'+2u_2'x=x^{\frac{5}{2}}$

$$\begin{cases} u_1'+u_2'x=0 \\ u_1'+2u_2'x=x^{\frac{5}{2}} \end{cases}， u_1'=-x^{\frac{5}{2}}, u_2'=x^{\frac{3}{2}}$$

$u_1=-\frac{2}{7}x^{\frac{7}{2}}, u_2=\frac{2}{5}x^{\frac{5}{2}}$， $y_p=-\frac{2}{7}x^{\frac{7}{2}}\cdot x+\frac{2}{5}x^{\frac{5}{2}}\cdot x^2=\frac{4}{35}x^{\frac{9}{2}}$

The general solution of (*) is $y=\frac{4}{35}x^{\frac{9}{2}}+c_1x+c_2x^2$

例 $(x-1)y''-xy'+y=(x-1)^2$ ...(*)

$(x-1)y''-xy'+y=0$ ...(**)

Given that $y_1=x, y_2=e^2$ are solutions of (**)

Find a particular solution of (*)，then find the general solution of (*)

Step 1 let $y_p=u_1x+u_2e^x$

$y_p'=(u_1+u_2x)+(u_1'x+u_2'e^x)$ and let $u_1'x+u_2'e^x=0$

$y_p''=u_1'+u_2'e^x+u_2e^x$ 代入(*)得

$u_1'+u_2'e^x = x-1$

$$\begin{cases} u_1'x+u_2'e^x = 0 \\ u_1'+u_2'e^x = x-1 \end{cases} \Rightarrow u_1' = -1, u_2' = xe^{-x} \Rightarrow u_1 = -x, u_2 = -(x+1)e^{-x}$$

$y_p = -x^2 - x - 1$

The general solution of (*) is $y = -x^2 - x - 1 + c_1 x + c_2 e^x$

例 $y''+3y'+2y = \dfrac{1}{1+e^x}$ p.260

$y = (e^{-x} + e^{-2x})\ln(1+e^x) + c_1 e^{-x} + c_2 e^{-2x}$

例 $(x^2-1)y''+4xy'+2y = \dfrac{2}{x+1} \ldots (*), y(0) = -1, y'(0) = -5$

Given that $y_1 = \dfrac{1}{x-1}, y_2 = \dfrac{1}{x+1}$ are solution of $(x^2-1)y''+4xy'+2y = 0 \ldots (**)$

$y = \dfrac{2\ln(x+1)}{x-1} + \dfrac{3x+1}{x^2-1}$

Exercises

5.7.4 $y''-2y'+2y = 3e^x \sec x \ldots (*)$

Step 1 $y''-2y'+2y = 0 \Rightarrow y_1 = e^x \cos x, y_2 = e^2 \sin x$

Step 2 let $y_p = u_1 e^x \cos x + u_2 e^x \sin x$

Step 3 $y_p' = (u_2 + u_1)e^x \cos x + (u_2 - u_1)e^x \sin x + (u_1'\cos x + u_2'\sin x)$

Let $u_1'\cos x + u_2'\sin x = 0$ then

$y_p' = (u_2 + u_1)e^x \cos x + (u_2 - u_1)e^x \sin x$

$y_p'' = (u_2'+u_1'+2u_2)e^x \cos x + (u_2'-u_1'-2u_1)e^x \sin x$

代入(*) 化簡 得 $-u_1'\sin x+u_2'\cos x=3\sec x$

Step 4 $\begin{cases} u_1'\cos x+u_2'\sin x=0 \\ -u_1'\sin x+u_2'\cos x=3\sec x \end{cases}$ 解出 $u_1'=-3\tan x, u_2'=3$

Step 5 $u_1=3\ln\left|\cos x\right|, u_2=3x$

$y_p=3e^x(\cos x\ln\left|\cos x\right|+x\sin x)$