§ $y''+p(x)y'+q(x)y=f(x)$

§ 5.6 Reduction of order (降階法)

$y''+p(x)y'+q(x)y=f(x)$

If we know y_1 is a solution of $y''+p(x)y'+q(x)y=0$ (complementary equatuon)，then let $y=uy_1$

可消去一項，得 $y_1u''+Q(x)u'=f(x)$ ，let $z=u'$ 得 z 的一階方程

例

假設已知 $y_1=e^2$ is a solution of $xy''-(2x+1)y'+(x+1)y=0 \cdots (**)$

Then solve the equation $xy''-(2x+1)y'+(x+1)y=x^2 \cdots (*)$

Let $y=ue^x$

Then $y'=u'e^x+ue^x$

$y''=u''e^x+2u'e^x+$

代入(*) 得 $(xu''-u')e^x=x^2$ let $z=u'$

$z'-\frac{z}{x}=xe^{-x}$ ，$z_1=x$ 是 $z'-\frac{z}{x}=0$ 的解，設 general solution $z=vx$

解出 $v=-e^{-x}+c_1$ then $u'=z=vx=-xe^{-x}+c_1x$

$u=(x+1)e^{-x}+\frac{1}{2}c_1x^2+c_2$

$y=ue^x=x+1+\frac{1}{2}c_1x^2e^x+c_2e^x$

Let $c_1=c_2=0$ $y_{p_1}=x+1$ is a solution of (*)

Let $c_1=2, c_2=0$ $y_{p_2}=x=x+1+x^2e^x$ is also a solution of (*)

$y_2=y_{p_2}-y_{p_1}=x^2e^x$ is a solution of (**)

$\{e^x, x^2e^x\}$ is a fundamentary set of solution (**)

例 Given that $y_1=x$ is a solution of the complementary equation

$x^2y''+xy'-y=0$ ...(*)

Find the general solution of $x^2y''+xy'-y=x^2+1$ ...(**)

Let y=ux then $x^3u''+3x^2u'=x^2+1$ ，let $z=u'$ then $x^3z'+3x^2z=x^2+1$

$(x^3z)'=x^2+1 \Rightarrow x^3z=\frac{1}{3}x^3+x+c_1$ ，$z=\frac{1}{3}+\frac{1}{x^2}+\frac{c_1}{x^3}$

$u=\frac{1}{3}x-\frac{1}{x}-\frac{1}{2x^2}c_1+c_2$

$y=ux=\frac{1}{3}x^2-1-\frac{c_1}{2x}+c_2x$

取 $c_1=0, c_2=1$ $y_{p_1}=\frac{1}{3}x^2-1=x$

取 $c_1=-2, c_2=1$ $y_{p_2}=\frac{1}{3}x^2-1+\frac{1}{x}+x$ then $y_{p_2}-y_{p_1}=\frac{1}{x}$ is a solution of (*)

We conclude that $y_1=x, y_2=\frac{1}{x}$ form a fundamental set of (*)