§ 5.4 The method of undetermined coefficients

$ay''+by'+cy=e^{\alpha x}G(x)$ ， where α is a constant and G(x) is a polynomial

30. **(a)** Prove that y is a solution of the constant coefficient equation

$$ay''+by'+cy=e^{\alpha x}G(x) \qquad \text{(A)}$$

if and only if $y=ue^{\alpha x}$, where u satisfies

$$au''+p'(\alpha)u'+p(\alpha)u=G(x) \qquad \text{(B)}$$

and $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation

$$ay''+by'+cy=0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x)=g_0+g_1x+g_2x^2+g_3x^3.$$

(b) Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p=A(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.4. Conclude that (A) has a particular solution of the form $y_p=e^{\alpha x}A(x)$.

(c) Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p=xA(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.5. Conclude that (A) has a particular solution of the form $y_p=xe^{\alpha x}A(x)$.

(d) Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p=x^2A(x)$, where A is a polynomial of the same degree as G, and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 5.4.6. Conclude that (A) has a particular solution of the form $y_p=x^2e^{\alpha x}A(x)$.

例 $y''-7y'+12y=4e^{2x}$

$y=2e^{2x}+c_1e^{3x}+c_2e^{4x}$

例 $y''-7y'+12y=5e^{4x}$

$y=5xe^{4x}+c_1e^{3x}+c_2e^{4x}$

例 $y''-8y'+16y=2e^{4x}$

$y''-8y'+16y=0 \quad y_1=e^{4x}, y_2=xe^{4x}$

Let $y=ue^{4x} \quad \Rightarrow u''=2 \quad y_p=x^2e^{4x}$

$y=e^{4x}(x^2+c_1+c_2x)$

例 $y''-3y'+2y=e^{3x}(-1+2x+x^2)$

Let $y=ue^{3x}$

例 $y''-7y'+12y=4e^{2x}+5e^{4x}$

$y_p=2e^{2x}+5xe^{4x}$

Exercises

1. $y''-6y'+8y=e^{x}(11-6x)$ $\qquad y=e^{x}(1-2x)+c_1e^{2x}+c_2e^{4x}$
2. $y''-4y'-5y=9e^{2x}(1+x), y(0)=0, y'(0)=-10$ $\qquad y=-e^{2x}(1+x)+2e^{-x}-e^{5x}$
3. $y''+4y'+3y=-e^{-x}(2+8x), y(0)=1, y'(0)=2$ $\qquad y=e^{-x}(2+x-2x^2)-e^{-3x}$
4. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$ find a particular solution

   $y''+y'+y=xe^{x}$，$y_{p_1}=-\frac{1}{3}e^{x}(1-x)$ $\qquad y''+y'+y=e^{-x}(1+2x)$，$y_{p_2}=e^{-x}(3+2x)$

   $y_p=y_{p_1}+y_{p_2}=\ldots$

38. Suppose $\alpha\neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\,dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y=\int e^{\alpha x}P(x)\,dx$$

with

$$P(x)=p_0+p_1x+\cdots+p_kx^k,\quad \text{(where } p_k\neq 0\text{)}.$$

(a) Show that $y=e^{\alpha x}u$, where

$$u'+\alpha u=P(x). \qquad \text{(A)}$$

(b) Show that (A) has a particular solution of the form

$$u_p=A_0+A_1x+\cdots+A_kx^k,$$

where $A_k, A_{k-1},\ldots,A_0$ can be computed successively by equating coefficients of $x^k, x^{k-1},\ldots,1$ on both sides of the equation

$$u_p'+\alpha u_p=P(x).$$

(c) Conclude that

$$\int e^{\alpha x}P(x)\,dx=\left(A_0+A_1x+\cdots+A_kx^k\right)e^{\alpha x}+c,$$

where c is a constant of integration.

§ $ay''+by'+cy=e^{\lambda x}(P(x)\cos\omega x+Q(x)\sin\omega x)$ where λ,ω are real numbers，$\omega\neq 0$ and P and Q are polynomials。

求原式一特別解

例 $y''-2y'+y=5\cos 2x+10\sin 2x$

Let $y_p=A\cos 2x+B\sin 2x$

A=1，B=-2

例 $y''+4y=8\cos 2x+12\sin 2x$

Let $y_p=x(Acox2x+B\sin 2x)$

A=-3，B=2

例 $y''+3y'+2y=(16+20x)\cos x+10\sin x$

Let $y_p=(A_0+A_1x)\cos x+(B_0+B_1x)\sin x$

$y_p=(1+2x)\cos x-(1-6x)\sin x$

例 $y''+y=(8-4x)\cos x-(8+8x)\sin x$

$y_p=x[(3+2x)\cos x+(2-x)\sin x]$

例 $y''+2y'+5y=e^{-x}[(6-16x)\cos 2x-(8+8x)\sin x]$ p.243

Let $y=ue^{-x}$