§ Riccati equation p.254 EX37

37. The nonlinear first order equation
$y'+y^2+p(x)y+q(x)=0...(1)$ is a Riccati equation，where $p(x)\neq 0, q(x)\neq 0$
If $q(x)=0$，the equation reduces to a Bernoulli equation。

(a) Show that y is a solution of (1) if and only if $y=\frac{z'}{z}$，where

$z''+p(x)z'+q(x)z=0 \quad \cdots(2)$

(b) Show that the general solution of(*) is $y=\frac{c_1z_1'+c_2z_2'}{c_1z_1+c_2z_2}\cdots(3)$

Where $\{z_1,z_2\}$ is a fundamental set of solution (2) and c_1,c_2 are arbitrary constants。

(c) Does the formula (3) imply that the first order equation (1)has a two-parameter family of solutions？

(a) $y=\frac{z'}{z}$，$y''=\frac{zz''-(z')^2}{z^2}$ 代入(1) 即得(2)

(b) $\{z_1,z_2\}$ is a fundamental set of solutuons of (2)，then $z=c_1z_1+c_2z_2$ is the general solution of (2) and $y=\frac{z'}{z}=\frac{c_1z_1'+c_2z_2'}{c_1z_1+c_2z_2}$ is the general solution of (1)

(c)

38. Find all solutions of the equations
(a) $y'+y^2+k^2=0$
(b) $y'+y^2-3y+2=0$
(c) $y'+y^2+5y-6=0$
(d) $y'+y^2+8y+7=0$
(e) $y'+y^2+14y+50=0$
(f) $6y'+6y^2-y-1=0$
(g) $36y'+36y^2-12y+1=0$

(b) Let $y=\frac{z'}{z}$ then $z''-3z'+2z=0\Rightarrow z=c_1e^x+c_2e^{2x}$

$$y=\frac{z'}{z}=\frac{c_1+2c_2e^x}{c_1+c_2e^x}$$

(e)Let $y=\frac{z'}{z}$ then z”+14z’+50z=0

$\lambda^2+14\lambda+50=0, \lambda=-7\pm i$

$z = c_1 e^{-7x} \cos x + c_2 e^{-7x} \sin x$

$$y = \frac{z'}{z} = -\frac{(7c_1 - c_2)\cos x + (c_1 + 7c_2)\sin x}{c_1 \cos x + c_2 \sin x}$$

39. Find all solution of the equation，given that y_1 is a solution

(a) $x^2(y' + y^2) - x(x+2)y + x + 2 = 0; y_1 = \dfrac{1}{x}$

(b) $y' + y^2 + 4xy + 4x^2 + 2 = 0; y_1 = -2x$

(c) $(2x+1)(y' + y^2) - 2y - (2x+3) = 0; y_1 = -1$

(d) $(3x-1)(y' + y^2) - (3x+2)y - 6x + 8 = 0; y_1 = 2$

(e) $x^2(y' + y^2) + xy + x^2 - \dfrac{1}{4} = 0; y_1 = -\tan x - \dfrac{1}{2x}$

(f) $x^2(y' + y^2) - 7xy + 7 = 0; y_1 = \dfrac{1}{x}$

(a) Let $y = \dfrac{z'}{z}$ then $x^2 z'' - x(x+2)z' + (x+2)z = 0$

$y_1 = \dfrac{1}{x} \Rightarrow z_1 = x$

Let $z = xu \Rightarrow u'' - u' = 0$，$u = c_1 + c_2 e^x$

$$y = \frac{z'}{z} = \frac{c_1 + c_2(e^x + xe^x)}{c_1 x + c_2 xe^x}$$

(b) Let $y = \dfrac{z'}{z}$ $z'' + 4xz' + (4x^2 + 2)z = 0$

$y_1 = -2x \Rightarrow z_1 = e^{-2x}$

Let $z = ue^{-2x} \Rightarrow u'' + (2x-1)u' = 0$

$\dfrac{u''}{u'} = 1 - 2x, \ln u' = x - x^2 + c_1$

$u' = e^{x - x^2 + c_1}$ $u = \int e^{x - x^2 + c_1} dx + c_2$

§ The nonlinear first order equation $y'+r(x)y^2+p(x)y+q(x)=0 \ldots(1)$ is the generalized Riccati equation

(a) Show that y is a solution of (1) if and only if $y=\dfrac{z'}{rz}$，where

$$z''+[p(x)-\frac{r'(x)}{r(x)}]z'+r(x)q(x)z=0 \ldots(2)$$

(b) Show that the general solution of (1) is $y=\dfrac{c_1z_1'+c_2z_2'}{r(c_1z_1+c_2z_2)}$，where $\{z_1,z_2\}$ is a fundamental set of solutions of (2) and c_1,c_2 are arbitrary constants。

(a) $y=\dfrac{z'}{rz}, y'=\dfrac{rzz''-(r'z+rz')z'}{r^2z^2}$ 代入仔細化簡得

$$z''+(p(x)-\frac{r'(x)}{r(x)})z'+r(x)q(x)z=0$$

(b) $z=c_1z_1+c_2z_2$ is a general solution of (B) then

$y=\dfrac{z'}{rz}=\dfrac{c_1z_1'+c_2z_2'}{r(c_1z_1+c_2z_2)}$ is the general solution of (A)