§ homogeneous equations

§ 5.2 $ay''+by'+cy=0$ where a、b、c are constant $a\neq 0$

1. $y''-y=0$ then $y=c_1e^x+c_2e^{-x}$

   If $y(0)=1, y'(0)=3$ then $y=2e^x-e^{-x}$

2. $y''+\omega y=0$ then $y=c_1\cos\omega x+c_2\sin\omega x$

   If $y(0)=1, y'(0)=3$ then $y=\cos\omega x+\dfrac{3}{\omega}\sin\omega x$

3. $x^2y''+xy'-4y=0$ then $y_1=x^2, y_2=\dfrac{1}{x^2}$ are solutions

   $y=c_1x^2+c_2\dfrac{1}{x^2}$ is a solution on $(-\infty,0)(0,\infty)$

4. The Wronskian and Abel formula

Theorem 5.1.3 *Suppose p and q are continuous on (a,b). Then a set $\{y_1, y_2\}$ of solutions of*

$$y''+p(x)y'+q(x)y=0 \tag{5.1.20}$$

on (a,b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a,b).

Theorem 5.1.4 *Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of*

$$y''+p(x)y'+q(x)y=0 \tag{5.1.27}$$

on (a,b), and define

$$W=y_1y_2'-y_1'y_2. \tag{5.1.28}$$

Let x_0 be any point in (a,b). Then

$$W(x)=W(x_0)e^{-\int_{x_0}^{x}p(t)\,dt}, \quad a<x<b. \tag{5.1.29}$$

Therefore either W has no zeros in (a,b) or $W\equiv 0$ on (a,b).

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of $\{y_1,y_2\}$ and (5.1.29) is Abel formula。

Then $W'+p(x)W=0$，that W is a solution of the initial problem $y'+p(x)y=0, y(x_0)=W(x_0)$

Theorem 5.1.5 *Suppose p and q are continuous on an open interval (a,b), let y_1 and y_2 be solutions of*

$$y''+p(x)y'+q(x)y=0 \tag{5.1.32}$$

on (a,b), and let $W=y_1y_2'-y_1'y_2$. Then y_1 and y_2 are linearly independent on (a,b) if and only if W has no zeros on (a,b).

1. $y''-y'-2=0$

$\lambda^2-\lambda-2=0, \lambda=-1,2$

$y_1=e^{-t}, y_2=e^{2t}$

$$W\{e^{-t},e^{2t}\}=\begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix}=3e^{t}\neq 0$$

$y=c_1e^{-t}+c_2e^{2t}$

Let $z=y'$，then $z'=y''=y'+2y=z+2y$

$$\begin{cases} y'=z \\ z'=2y+z \end{cases} \quad A=\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \lambda=-1,2$$

$$\lambda_1=-1, \xi^1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}; \lambda_2=2, \xi^2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$X^1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}, X^2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}$$ general solution

$$X=c_1X^1+c_2X^2=c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}+c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}$$ 所以 $y=c_1e^{-t}+c_2e^{2t}$

2. $y''-2y'+2y=0$

$\lambda^2-2\lambda+2=0, \lambda=1\pm i$

$y_1=e^{t}\cos t, y_2=e^{t}\sin t$

Solve the initial problem if y(0)=3， $y'(0)=-2$

3. $y''-2y'+y=0$

$\lambda^2-2\lambda+1=0, \lambda=1,1$

$y_1=e^{t}, y^2=te^{2t}$

4. Verify that $y_1=\dfrac{1}{x-1}, y_2=\dfrac{1}{x+1}$ are solutions of $(x^2-1)y''+4xy'+2y=0$ on $(-\infty,-1),(-1,1),(1,\infty)$

What is the general solution on each intervals？

Abel formula $W(x)=W(x_0)e^{-\int_{x_0}^{x}p(t)dt}$

5. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0,$$

given that $W(0)=1$. (This is *Legendre's equation*.)

$$W(x)=\frac{1}{1-x^2}$$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y''+xy'+(x^2-\nu^2)y=0,$$

given that $W(1)=1$. (This is *Bessel's equation*.)

$$W(x)=\frac{1}{x}$$

7.

9. (This exercise shows that if you know one nontrivial solution of $y''+p(x)y'+q(x)y=0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y''+p(x)y'+q(x)y=0 \qquad \text{(A)}$$

that has no zeros on (a,b). Let $P(x)=\int p(x)\,dx$ be any antiderivative of p on (a,b).

(a) Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1y_2'-y_1'y_2=Ke^{-P(x)} \qquad \text{(B)}$$

on (a,b), then y_2 also satisfies (A) on (a,b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on (a,b).

(b) Conclude from **(a)** that if $y_2=uy_1$ where $u'=K\dfrac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a,b).

8. 假設已知 $y''-2y'-3y=0$ 有一解 $y_1=e^{3x}$，利用 Abel 公式求另一解 p.205 ex10

假設 $y_2=uy_1$ 則 $W(x)=y_1y_2'-y_1'y_2=u'y_1^2$

令 p(x)=-2，$P(x)=\int p(x)dx=-2x$，$u'(e^{3x})^2=Ke^{-P(x)}=Ke^{2x}$

$u'=Ke^{-4x}$，$u=-\dfrac{K}{4}e^{-4x}$ 取 K=-4，$u=e^{-4x}, y_2=uy_1=e^{-x}$

9. $y''-2ay'+a^2y=0$ ，a=constant， $y_1=e^{ax}$ is a solution ex 12

$y_2=xe^{ax}$

10. $x^2y''-xy'+y=0, y_1=x$ ex14

$y_2=x\ln x$

11. $x^2y''-2xy'+(x^2+2)y=0, y_1=x\cos x$ ex18

$y_2=x\sin x$

12. $(3x-1)y''-(3x+2)y'-(6x-8y)=0, y_1=e^{2x}$ ex20

$y_2=xe^{-x}$