

§ Homogeneous nonlinear equations

$$y' = f\left(\frac{y}{x}\right)$$

例 $xy' - y = xe^{-\frac{y}{x}}$ ($y' = \frac{y}{x} + e^{-\frac{y}{x}}$)

Let $y=ux$, $y' = u'x + u$

$$u'x + u = u + e^{-u}$$

$$e^u u' = \frac{1}{x}$$

$$e^u = \ln|x| + c$$

$$y = x \ln(\ln|x| + c)$$

例 $x^2 y' = y^2 + xy - x^2 \dots (*)$, $y(1) = 2$

Let $y=ux$

$$y' = u'x + u = \frac{y^2 + xy - x^2}{x^2} = u^2 + u - 1$$

$$\frac{u'}{u^2 - 1} = \frac{1}{x}$$

...

$$u = \frac{1 + cx^2}{1 - cx^2}$$

$$y = \frac{x(1 + cx^2)}{1 - cx^2} \dots (**)$$
 $c=0$ 得 $y=x$

$Y=-x$ 也滿足(*) 但是無法從(**)得到, 所以(*)的 general solution 是 $y=-x$ and

$$y = \frac{x(1 + cx^2)}{1 - cx^2}$$

$$y(1) = 2 \Rightarrow c = \frac{1}{3}$$

Exercises

1. $y' - 2y = xy^3 \dots (*)$, $y(0) = 2\sqrt{2}$

解(1) Bernoulli equation let $z = y^{1-3} = y^{-2}$ 解(2) $y_1 = e^{2x}$ let $y = ue^{2x}$

$$y = \sqrt{\frac{1}{-\frac{1}{2}x + \frac{1}{8} + c}}, \quad c=0$$

2.