§ Integrating factor

1. $\frac{dy}{dx}+p(x)y=q(x)$ 有積分因子 $\mu(x)=\exp\int p(x)dx$

例 $y'+\frac{3}{t}y=\frac{e^t}{t^2}$ ，y(1)=2

$\mu(t)=\exp\int\frac{3}{t}dt=e^{3\ln t}=t^3,t>0$

$(\mu(t)y(t))'=\mu(t)\frac{e^t}{t^2}=te^t$

$t^3y=\int te^t dt=(t-1)e^t+c$

$y(t)=\frac{(t-1)e^t+2}{t^3}$

習作

(1) $xy'+2y=x^2-x+1,x>0,y(1)=\frac{1}{2}$ $\qquad$ $y=\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}+\frac{1}{12x^2}$

(2) $(x^2-x)y'+(1-2x)y=-x^2$ $\qquad$ $y=x+c(x^2-x)$

(3) $y'-y=2xe^{2x},y(0)=1$ $\qquad$ $y=2(x-1)e^{2x}+3e^{x}$

例 $y'=\frac{1}{e^y-x},y(1)=0$

$\frac{dx}{dy}=e^y-x$

$\frac{dx}{dy}+x=e^y$ ...

$xe^y=\int e^{2y}dy=\frac{1}{2}e^{2y}+c,c=\frac{1}{2}$

$y=\cosh^{-1}x$

Mdx+Ndy=0

2. 若 $\frac{\mathrm{M}_y - N_x}{N} = f(x)$ 則有積分因子 $\mu = \exp\int f(x)dx$

若 $\frac{M_y - N_x}{M} = f(y)$ 則有積分因子 $\mu = \exp\int f(y)dy$

例 $2xydx + (y^2 - x^2)dy = 0$

$\frac{N_x - M_y}{M} = -\frac{2}{y}$ ， $\mu(y) = \exp\int -\frac{2}{y}dy = \frac{1}{y^2}$

$\frac{2x}{y}dx + \frac{y^2 - x^2}{y^2}dy = 0$

$\begin{cases} \frac{\partial\varphi}{\partial x} = \frac{2x}{y} \\ \frac{\partial\varphi}{\partial y} = 1 - \frac{x^2}{y^2} \end{cases}$ $\Rightarrow$ general solution $\frac{x^2}{y} + y = c$

3. 若 $\frac{\mathrm{N}_x - M_y}{xM - yN} = R(xy)$ 則有積分因子 $\mu = \exp\int R$

例 $(3x + \frac{6}{y})dx + (\frac{x^2}{y} + \frac{3y}{x})dy = 0$

$\frac{\mathrm{N}_x - M_y}{xM - yN} = \frac{1}{xy}$ ， $\mu = \exp\int R = xy$

$(3x^2 y + 6x)dx + (x^3 + 3y^2)dy = 0$ 解出 $\varphi(x, y) = x^3 y + 3x^2 + y^3 + c$

4. 若 M d x + N d y=0 是 homogeneous equation 則有積分因子 $\mu = \frac{1}{xM + yN}$

例 (x-2y) dx +(2x-y) dy=0

解 1 $\mu = \frac{1}{x(x-2y) + y(2x-y)} = \frac{1}{x^2 - y^2}$

$\frac{x - 2y}{x^2 - y^2}dx + \frac{2x - y}{x^2 - y^2}dy = 0$

$\begin{cases} \frac{\partial\varphi}{\partial x} = \frac{x - 2y}{x^2 - y^2} \ldots(1) \\ \frac{\partial\varphi}{\partial y} = \frac{2x - y}{x^2 - y^2} \ldots(2) \end{cases}$

$$\frac{x-2y}{x^2-y^2}=-\frac{1}{2}\frac{1}{x+y}+\frac{3}{2}\frac{1}{x-y}$$

由(1) $\varphi=-\frac{1}{2}\ln|x+y|+\frac{3}{2}\ln|x-y|+h(y)$ 代入(2)

得 $h'(y)=\frac{-4x}{y^2-x^2}=\frac{2}{y+x}-\frac{2}{y-x}$ ，$h(y)=2\ln|y+x|-2\ln|y-x|$

$$\varphi(x,y)=-\frac{1}{2}\ln|y+x|+\frac{3}{2}\ln|y-x|+2\ln|y+x|-2\ln|y-x|$$

$$|y-x|=c^2|y+x|^3$$

解 2. Let y=vx

$$\frac{dy}{dx}=\frac{2y-x}{2x-y}=\frac{2(\frac{y}{x})-1}{2-(\frac{y}{x})}=\frac{2v-1}{2-v}$$

$\frac{dy}{dx}=xv'+v=\frac{2v-1}{2-v}$, $xv'=\frac{2v-1}{2-v}-v=\frac{v^2-1}{2-v}$

$\frac{dx}{x}=(\frac{2-v}{v^2-1})dv$, $\frac{2-v}{v^2-1}=-\frac{3}{2}\frac{1}{v+1}+\frac{1}{2}\frac{1}{v-1}$

$$\ln|x|+\ln|c|=-\frac{3}{2}\ln|v+1|+\frac{1}{2}\ln|v-1|$$

...最後還是得到 $|y-x|=c^2|y+x|^3$

§ P dx + Q dy=0...(*) 定理 Lie 1874

$$(x,y)\xrightarrow{G}(\hat{x}(t),\hat{y}(t)),(\hat{x}(0),\hat{y}(0))=(x,y)$$

其中 G 是一個 Lie group(或者是 1-parameter group φ_t)

$$(\xi,\eta)=\frac{d}{dt}(\hat{x}(t),\hat{y}(t))\Big|_{t=0}$$

若 G 保持(*)不變 則 $\mu=\frac{1}{\xi P+\eta Q}$ 是(*)的一個積分因子

而 $X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$ 是 G 的無窮小生成元(infinitesimal generator)

平面的旋轉群顯然是一個 Lie group

$\varphi_t:(x,y)\to(x\cos t-y\sin t, x\sin t+y\cos t)$

$\dfrac{dy}{dx}=\dfrac{y+x(x^2+y^2)}{x-y(x^2+y^2)}$ ...(*)　寫成 $[-y-x(x^2+y^2)]dx+[x-y(x^2+y^2)]dy=0$

上述旋轉群使得(*)不變

(maps each integral curves into another integral curves，in other words leaves the equation (*) stable.)

Let X=x cost-y sin t，Y=x sin t+ y cost

則 X d X+Y d Y=x d x + y d y

若令 X=R cos A , Y=R sin A 則-Y d X + X d Y=R^2dA

同理 令 x=r cos a ,y=r sin a 則-y d x +x d y=r^2da

R=r ，A=a + t 則 d A=da (t 是常數)

所以[-y-x(x^2+y^2)] dx + [x-y(x^2+y^2)] dy = -y dx +x dy - (x^2+y^2)(x dx+y dy) is invariant if we replace x、y by X、Y.

$\dfrac{d}{dt}\varphi_t\big|_{t=0}=(-y,x)$ ，則 $\mu=\dfrac{1}{x^2+y^2}$ 是積分因子

(*)變成 $(\dfrac{-y}{x^2+y^2}-x)dx+(\dfrac{x}{x^2+y^2}-y)dy=0$

解為 U=c

則 $\dfrac{\partial U}{\partial x}=\dfrac{-y}{x^2+y^2}-x$...(1), $\dfrac{\partial U}{\partial y}=\dfrac{x}{x^2+y^2}-y$...(2)

由(1) $U=\int\dfrac{-y}{x^2+y^2}dx-\dfrac{1}{2}x^2+h(y)=\arctan(\dfrac{y}{x})-\dfrac{1}{2}x^2+h(y)$ 代入(2)

$\dfrac{\partial U}{\partial y}=\dfrac{\frac{1}{x}}{1+(\frac{y}{x})^2}+h'(y)$ ，則 $h'(y)=-y, h(y)=-\dfrac{1}{2}y^2$

所以 $U=\arctan\dfrac{y}{x}-\dfrac{1}{2}(x^2+y^2)$ ，原方程式的解為 $\arctan\dfrac{y}{x}-\dfrac{1}{2}(x^2+y^2)=c$

解 $\dfrac{dy}{dx}=\dfrac{y+x(x^2+y^2)}{x-y(x^2+y^2)}$

$xdx+ydy+\dfrac{ydx-xdy}{x^2+y^2}=0$ 兩邊積分

$\dfrac{1}{2}(x^2+y^2)+\arctan(\dfrac{x}{y})=c$

§ homogeneous equation $\frac{dy}{dx} = f(\frac{y}{x})$

1. $-f(\frac{y}{x})dx + dy = 0$

$(x,y) \overset{\varphi_t}{\rightarrow} (e^t x, e^t y)$ ， $(e^0 x, e^0 y) = (x,y)$

$\frac{d\varphi_t}{dt}\Big|_{t=0} = (x,y)$ ， $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

積分因子為 $(-xf(\frac{y}{x}) + y)^{-1}$

2. Let y=ux

例 $x + yy' = 2y$

$y' = 2 - \frac{x}{y}$ ，let y=u x then $y' = u'x + u$

$u'x + u = 2 - \frac{1}{u}$

$u'x = -u + 2 - \frac{1}{u} = \frac{-(u-1)^2}{u}$ ， $\frac{udu}{-(u-1)^2} = \frac{dx}{x}$

…

$(y-x)e^{\frac{x}{x-y}} = c$

習作

1. $(x+y) + (x-y)y' = 0$

$x^2 + 2xy - y^2 = c$

2. $(2x - y + 3) - (x - 2y + 3)y' = 0$

平移消掉常數項

$u = x+1, v = y-1$ ， $(2u - v) - (u - 2v)\frac{dv}{du} = 0$

$x^2 - xy + y^2 + 3x - 3y = c$

例 解 $xy'+3y=\dfrac{2}{x(1+x^2)},\ y(-1)=0$

解(1) $\mu(x)=\exp\int\dfrac{3}{x}dx=x^3$

$(x^3y)'=\dfrac{2x}{1+x^2}\quad\cdots\quad y=\dfrac{1}{x^3}\ln(\dfrac{1+x^2}{2})$

解(2) $y'+\dfrac{3y}{x}=\dfrac{2}{x^2(1+x^2)}\ldots(*)$

考慮 complementary eq $xy'+3y=0\quad y_1=x^{-3}$

Let $y=ux^{-3}$ is the general solution 代入(*) 得

$u'x^{-3}=\dfrac{2}{x^2(1+x^2)}$

$u=\int\dfrac{2x}{1+x^2}dx=\ln(1+x^2)+c$

例 $(3xy+6y^2)dx+(2x^2+9xy)dy=0$

$M_y-N_x=-x+3y$

假設 $\mu(x,y)=P(x)Q(y)$ 是積分因子 P(x)Q(y)Mdx+P(x)Q(y)Ndy=0 is exact

Then $\dfrac{\partial}{\partial y}(PQM)=\dfrac{\partial}{\partial x}(PQN)$

$\cdots$

$M_y-N_x=\dfrac{P'(x)}{P(x)}N-\dfrac{Q'(y)}{Q(y)}M$ let $p(x)=\dfrac{P'(x)}{P(x)},q(y)=\dfrac{Q'(y)}{Q(y)}$

$-x+3y=p(x)(2x^2+9xy)-q(y)(3xy+6y^2)=xp(x)(2x+9y)-yq(y)(3x+6y)$

Let xp(x)=A，yq(y)=B 解出 A=B=1

P(x)=x，Q(y)=y

$\mu(x,y)=xy$

$\Downarrow$

$x^3y^2+3x^2y^2=c$

Exercises

1. $-ydx+(x+x^6)dy=0$ $\qquad y=cx(1+x^5)^{-\frac{1}{5}}$

2. $y\sin ydx+x(\sin y-y\cos y)dy=0$ $\qquad \mu(x,y)=x^{-2}y^{-2},\varphi(x,y)=\dfrac{-\sin y}{xy}+c$