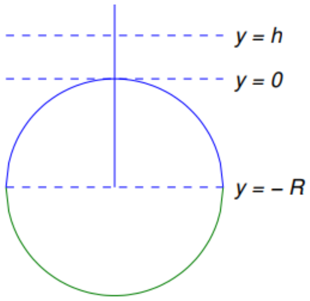
§ Escape velocity p.159 W.F.Trench

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth，where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected。

Let t=0 be the time when burnout occurs。



$F=-\dfrac{K}{(y+R)^2}$ where R is the Earth's radius

When y=0，F=-mg，$K=mgR^2$

$$F=m\frac{d^2y}{dt^2}=\frac{-mgR^2}{(y+R)^2}$$

$$\frac{d^2y}{dt^2}=\frac{-gR^2}{(y+R)^2} \quad ...(*)$$

We'll show that there's a number v_e, called the *escape velocity*, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t) = \infty$ in this case? For a proof, see Exercise 20.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

$$v=y',\frac{d^2y}{dt^2}=\frac{dv}{dt}=\frac{dv}{dy}\frac{dy}{dt}=v\frac{dv}{dy}$$

(*)變成 $v\dfrac{dv}{dy}=\dfrac{-gR^2}{(y+R)^2}$，When t=0，$v(h)=v_0$

兩邊對 y 積分，$\dfrac{v^2}{2}=\dfrac{gR^2}{y+R}+c$，since $v(h)=v_0$，$c=\dfrac{{v_0}^2}{2}-\dfrac{gR^2}{h+R}$

Then $\dfrac{v^2}{2}=\dfrac{gR^2}{y+R}+\dfrac{{v_0}^2}{2}-\dfrac{gR^2}{h+R}$...(**)

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in (4.3.23) is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

19. **(a)** Show that Eqn. (4.3.23) can be rewritten as

$$v^2 = \frac{h - y}{y + R} v_e^2 + v_0^2.$$

(b) Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1 - \rho^2}.$$

(c) By requiring that $v(y_m) = 0$, use Eqn. (4.3.22) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1 - \rho^2)(y_m - y)}{y + R} \right]^{1/2},$$

where y_m and ρ are as defined in **(b)** and $y \geq h$.

(d) Deduce from **(c)** that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.