Lesson 6 Extrema of Functions of two Variables

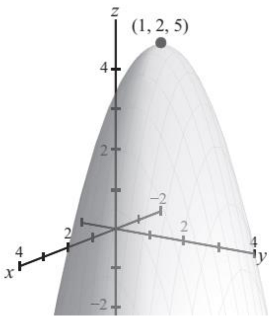
Second partials test: Let (a, b) be a critical point of f.

Define the quantity $d = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$. Then, we have the following.

1. $d > 0, f_{xx}(a, b) > 0 \Rightarrow$ relative minimum.
2. $d > 0, f_{xx}(a, b) < 0 \Rightarrow$ relative maximum.
3. $d < 0, \Rightarrow$ saddle point.
4. $d = 0$: Test is inconclusive.

Example

1.



If possible，find the highest and lowest points on the graph of the function $f(x, y) = 2x + 4y - x^2 - y^2$

$\frac{\partial f}{\partial x} = 2 - 2x$ ，$\frac{\partial f}{\partial y} = 4 - 2y$ ，critical point (x,y)=(1,2) is a maximum。

$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$

$d = f_{xx}(1,2) f_{yy}(1,2) - (f_{xy}(1,2))^2 = 4 > 0$ ， $f_{xx}(1,2) < 0$

Hence (1，2) is a relative maximum

2. $z = f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$ ，find the critical points

$$\begin{cases} \frac{\partial f}{\partial x} = -6x + 6y = 0 \\ \frac{\partial f}{\partial y} = 6y - 6y^2 + 6x = 0 \end{cases}$$ ，(0,0),(2,2) are the critical points

$$\frac{\partial^2 f}{\partial x^2} = -6, \frac{\partial^2 f}{\partial x \partial y} = 6, \frac{\partial^2 f}{\partial y^2} = 6 - 12y$$

$D = \begin{vmatrix} -6 & 6 \\ 6 & 6 - 12y \end{vmatrix}$ ，D(0,0)<0，D(2,2)>0， $f_{xx}(2,2) < 0$

(0,0) is a saddle point，and (2,2) is a relative maximum

Exercises

9. Find the critical point(s) of the function $f(x, y) = x^{2/3} + y^{2/3}$ and determine the relative extrema.

10. An open box is to be constructed with 2 square meters of material. Determine the dimensions of the box so that the volume is a maximum.

11. Verify that that the partial derivative with respect to x for $V = xy\left[\dfrac{C-3xy}{4(x+y)}\right]$ is $V_x = \dfrac{y^2}{4(x+y)^2}\left(C-3x^2-6xy\right)$.

12. Verify that $V_x = \dfrac{y^2}{4(x+y)^2}\left(C-3x^2-6xy\right)=0$ and $V_y = \dfrac{x^2}{4(x+y)^2}\left(C-3y^2-6xy\right)=0$ gives the solution $x = y = 12$.