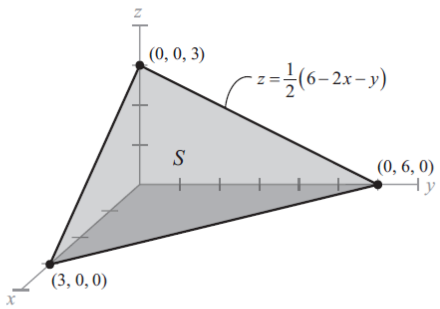
Lesson 34

Let the surface S be given by $z = g(x, y)$, and let $f(x, y, z)$ be defined at all points on S. Let R be the projection of S onto the xy-plane. With suitable hypotheses on f and g, the **surface integral** is

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+(g_x)^2+(g_y)^2}\,dA.$$

§ Surface Integral and Flux

Example



$$\iint_S (y^2+2yz)\,dS =$$

$$\varphi(x,y,z) = (x, y, 3-x-\frac{1}{2}y)$$

$$\varphi_x = (1,0,-1), \varphi_y = (0,1,-\frac{1}{2})$$

$$E = \varphi_x\cdot\varphi_x = 2, F = \varphi_x\cdot\varphi_y = \frac{1}{2}, G = \varphi_y\cdot\varphi_y = \frac{5}{4}$$

$$\iint_S (y^2+2yz)\,dS = \int_0^3\int_0^{6-2x}(6y-2x^2)\times\frac{3}{2}\,dydx = \ldots = \frac{243}{2}$$

Example

Let S be the portion of the paraboloid $z = g(x, y) = 4 - x^2 - y^2$ lying above the xy-plane and oriented by an upward unit normal.

A fluid is flowing through the surface according to the vector field F(x,y,z)=(x,y,z)

Find $\iint_S F\cdot N\,dS$ the rate of mass flow through the surface

$$G(x,y,z) = z - g(x,y) = z-(4-x^2-y^2)$$
$$NdS = \nabla G dA = (2x, 2y, 1)dA$$
$$\iint_S F\cdot N\,dS = \iint_R (2x^2+2y^2+z)dA = \iint_R (x^2+y^2+4)dA = \ldots = 24\pi$$

Exercise

5. Set up the integral in polar coordinates for the surface integral $\iint_S \frac{xy}{z}\,dS$, where S is the surface $z = x^2 + y^2$, $4 \le x^2 + y^2 \le 16$.

6. Set up the flux integral $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$ and S is the surface $z = 1 - x - y$ in the first octant.

7. Use polar coordinates to set up the flux integral $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface $x^2 + y^2 + z^2 = 36$ in the first octant.

Lesson 35

§ Divergence Theorem

Let Q be a solid region in space bounded by the surface S oriented outward by a unit normal $\mathbf{N}$. Let $\mathbf{F}(x, y, z)$ be a vector field whose component functions have continuous first partial derivatives. The **divergence theorem** states that

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV.$$

If div$\mathbf{F} > 0$, we say that we have a **source**. If div$\mathbf{F} < 0$, we have a **sink**.

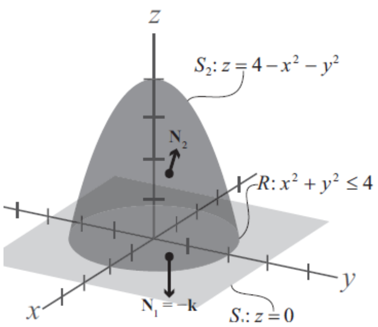
Gauss's law relates the flux out of a surface to the total charge inside the surface.

In particular, if E is an electric field, then $\iint_S \mathbf{E}\cdot\mathbf{N}\,dS = \frac{Q}{\varepsilon_0}$.

Here, Q is the electric charge inside a sphere, and ε_0 is the permittivity of space, or the electric constant.

Example

Let Q be the solid region between the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Verify the divergence theorem for the function $\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$. (See **Figure 35.1**.)



Let's first do the flux integral. We have two surfaces to consider. The flat plane $S_1: x^2 + y^2 \le 4$ has outward unit normal $\mathbf{N}_1 = -\mathbf{k}$.

So，for the surface

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}_1\,dS = \iint_{S_1} \mathbf{F}\cdot(-\mathbf{k})\,dS = \iint_R \left[2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}\right]\cdot(-\mathbf{k})\,dA = \iint_R \left(-y^2\right)dA.$$

$$= -4\pi$$

The paraboloid forms the second surface

$$S_2 : z = 4 - x^2 - y^2 \text{ ， } G = z - (4 - x^2 - y^2)$$

$$N_2 = \frac{\nabla G}{\|\nabla G\|} = \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\mathbf{N}_2 dS = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\sqrt{(g_x)^2 + (g_y)^2 + 1}\,dA$$

$$= \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}\sqrt{4x^2 + 4y^2 + 1}\,dA = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA.$$

So, $\iint_{S_2} \mathbf{F} \bullet \mathbf{N}_2 \, dS = \iint_R (2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}) \bullet (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, dA = \iint_R (4xz + 2xy + y^2) \, dA.$

$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy + y^2) dxdy = 4\pi$$

$$\iint_S \mathbf{F} \bullet \mathbf{N} \, dS = \iint_{S_1} \mathbf{F} \bullet \mathbf{N}_1 \, dS + \iint_{S_2} \mathbf{F} \bullet \mathbf{N}_2 \, dS = -4\pi + 4\pi = 0.$$

另解

$$divF = \frac{\partial \mathrm{F}_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial \mathrm{F}_z}{\partial z} = 0$$

$$\iint_S \mathbf{F} \bullet \mathbf{N} \, dS = \iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q 0 \, dV = 0.$$

Exercise

4. Use the divergence theorem to evaluate the flux integral $\iint_S \mathbf{F} \bullet \mathbf{N} \, dS$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

5. Use the divergence theorem to evaluate the flux integral $\iint_S \mathbf{F} \bullet \mathbf{N} \, dS$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$ and S is the surface bounded by $x^2 + y^2 = 25$, $z = 0$, and $z = 7$.

6. For the constant vector field $\mathbf{F}(x, y, z) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, verify that $\iint_S \mathbf{F} \bullet \mathbf{N} \, dS = 0$, where V is the volume of the closed surface S.