Lesson 31

§ Green theorem

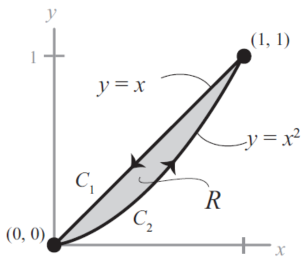
**Green's theorem**: Let $C$ be a piecewise smooth closed curve, oriented counterclockwise. The curve is traversed once with the simply connected region $R$ on its left. Then,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Examples

1. let R be the region in the first quadrant bounded by the graphs of y=x and $y = x^2$

   Let C be the boundary，oriented counterwise。




$$\int_C y^2dx + x^2dy =$$

$$\int_C y^2dx + x^2dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_0^1\int_{x^2}^{x}(2x-2y)dydx = \frac{1}{30}$$

2.

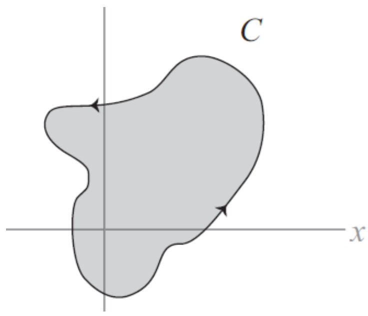
A particle travels once counterclockwise around the circle of radius 3 centered at the origin, subject to the force $\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$. Use Green's theorem to find the work done by the force field.

$$W = \int_C y^3dx + (x^3+3xy^2)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \iint_R 3x^2dA$$

Let $x = r\cos\theta, y = r\sin\theta$

$$\iint_R 3x^2dA = \int_0^{2\pi}\int_0^3 3(r\cos\theta)^2 rdrd\theta = \frac{243\pi}{4}$$

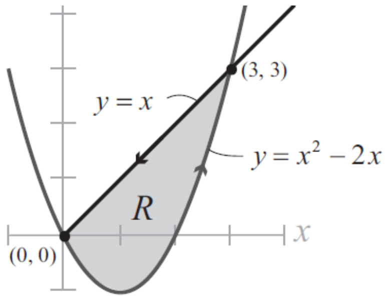
3.




Calculate $\int_C y^3dx + 3xy^2dy =$

0

4. Use Green's theorem to evaluate the integral $\int_C (y-x)\,dx+(2x-y)\,dy$, where $C$ is the boundary of the region between the graphs of $y=x$ and $y=x^2-2x$.




$$\int_C (y-x)dx+(2x-y)dy=$$

$$\int_C (y-x)dx+(2x-y)dy=\iint_R dA$$

$$=\int_0^3\int_{x^2-2x}^{x} dydx=\frac{9}{2}$$

Exercises

**4.** Use Green's theorem to evaluate the line integral $\int_C e^x \cos 2y\,dx-2e^x \sin 2y\,dy$, where $C$ is the circle $x^2+y^2=9$.

**5.** Find the work done by the force $\mathbf{F}(x, y)=xy\mathbf{i}+(x+y)\mathbf{j}$ that is moving a particle counterclockwise once around the unit circle $x^2+y^2=1$.

**6.** Find the work done by the force $\mathbf{F}(x, y)=\left(x^{3/2}-3y\right)\mathbf{i}+\left(6x+5\sqrt{y}\right)\mathbf{j}$ that is moving a particle counterclockwise once around the triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 5)$.

**5.** Let $C$ be the line segment joining the points $(x_1, y_1)$ and $(x_2, y_2)$.

Verify the formula $\int_C x\,dy-y\,dx=x_1y_2-x_2y_1$.

**6.** Find the area enclosed by the hexagon with vertices $(0, 0)$, $(2, 0)$, $(3, 2)$, $(2, 4)$, $(0, 3)$, $(-1, 1)$.

**7.** Prove that $\int_C f(x)\,dx+g(y)\,dy=0$ if $f$ and $g$ are differentiable functions and $C$ is a piecewise smooth simple closed path.