Lesson 30

§ Fundamental theorem of line integrals

(1) F is conservative　(2) line integral is path independent　(3) $\oint_C F\cdot dr=0$

(1) (2) (3)等價

Example

$F(x,y)=(2xy,x^2-y)$，C：連接(-1,4) (1,2)的曲線

求 $\int_C F\cdot dr=$

$M=2xy, N=x^2-y$ 則 $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}=2x$ 所以 F is conservative

Let $F=\nabla f=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$，可解出 $f(x,y)=x^2y-\frac{1}{2}y^2$

$\int_C F\cdot dr=\int_C \nabla f\cdot dx=f(1,2)-f(-1,4)=4$

Example

$F(x,y,z)=(2xy,x^2+z^2,2yz)$，C：連接(1,1,0) (0,2,3)的曲線

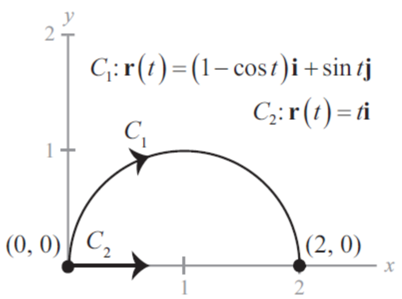
求 $\int_C F\cdot dr=$

$$curF=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+z^2 & 2yz \end{vmatrix}=(0,0,0)$$

The potential function $f(x,y,z)=x^2y+yz^2$

$\int_C F\cdot dr=\int_C \nabla f\cdot dx=f(0,2,3)-f(1,1,0)=17$

Example



$F(x,y)=(y^3+1,3xy^2+1)$

求 $\int_C F\cdot dr=$

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Exercises

6. Evaluate the line integral $\int_C yz\,dx + xz\,dy + xy\,dz$, where C is the curve $\mathbf{r}(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4$.

7. Evaluate the line integral $\int_C yz\,dx + xz\,dy + xy\,dz$, where C is the curve $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 2$.

8. Evaluate the line integral $\int_C \cos x \sin y\,dx + \sin x \cos y\,dy$, where C is the line segment from $(0, -\pi)$ to $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$.

9. Find the work done by the force field $\mathbf{F}(x, y) = \frac{2x}{y}\mathbf{i} - \frac{x^2}{y^2}\mathbf{j}$ in moving an object from the point $(-1, 1)$ to the point $(3, 2)$.

10. Verify that $\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$ is conservative.

What is the value of the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ if C is the circle $x^2 + (y - 4)^2 = 1$?