Lesson 29 Line Integrals and Works

- Let C be a planar curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$. If $\mathbf{F}$ is a vector field with unit tangent vector $\mathbf{T}$, then the **line integral** of $\mathbf{F}$ is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\left(x(t), y(t)\right)\cdot\mathbf{r}'(t)\,dt.$$

- Line integrals in **differential form**: If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy$.

Example

1. Evaluate the line integral $\int_C y\,dx + x^2 dy$, where C is the parabola $y = 4x - x^2$ from $(4, 0)$ to $(1, 3)$.

$$\frac{69}{2}$$

2. A spring r(t)=(cost,sint,t) is a circular helix $0 \le t \le 6\pi$, density $\rho = 1 + z$

   Then the mass of the spring is $\int_\gamma \rho ds$

$$r'(t) = (-\sin t, \cos t, 1) \text{ , } ds = |r'(t)|\,dt = \sqrt{2}dt$$

$$\int_0^{6\pi} (1+t)\sqrt{2}dt = 6\sqrt{2}\pi(3\pi + 1)$$

3.

Exercises

7. Evaluate the line integral $\int_C (3y - x)\,dx + y^2\,dy$, where C is the path given by $x = 2t$, $y = 10t$, $0 \le t \le 1$.

8. Evaluate the line integral $\int_C \left(x + 3y^2\right)dy$, where C is the path given by $x = 2t$, $y = 10t$, $0 \le t \le 1$.