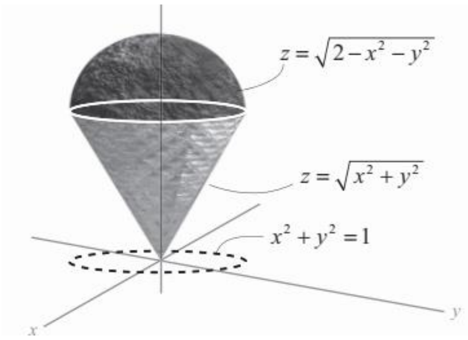
Lesson 26 Triple Integrals in Spherical Coordinates

- Let $P = (x, y, z)$ be a point in space. Its **spherical coordinates** are (ρ, θ, ϕ), where ρ is the distance from P to the origin, θ is the same angle as used in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment $\overline{OP}$, $0 \le \phi \le \pi$.

$$r = \rho \sin\phi, x = r\cos\theta = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

- The **differential of volume** in spherical coordinates is $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

Example



Find the volume of the ice-cream cone bounded above by the upper half of the sphere $x^2 + y^2 + z^2 = 2$ and below by $z = \sqrt{x^2 + y^2}$

Converting to spherical coordinates,

$$x^2 + y^2 + z^2 = 2 = \rho^2 \Rightarrow \rho = \sqrt{2} \text{ and } z = \rho\cos\phi \Rightarrow 1 = \sqrt{2}\cos\phi \Rightarrow \cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}.$$

Also, $0 \le \theta \le 2\pi$. The ice-cream cone is given by $0 \le \rho \le \sqrt{2}, 0 \le \phi \le \frac{\pi}{4}, 0 \le \theta \le 2\pi$. The volume is

$$V = \iiint_Q dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{4\pi}{3}\left[\sqrt{2} - 1\right].$$

Exercises

6. Convert the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2}^{2+\sqrt{4-x^2-y^2}} x \, dz \, dy \, dx$ to spherical coordinates.