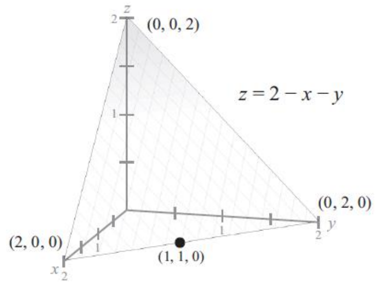
§ Lesson 23 Surface Area

For a surface given by $z = f(x, y)$ defined over a region R in the xy-plane, the **surface area** is

$$S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

Example 1



z=2-x-y 在第一卦限的面積

解 1

$$X(x,y) = (x, y, 2-x-y)$$

$$X_x = (1,0,-1), X_y = (0,1,-1)$$

$$E = X_x \cdot X_x = 2, F = X_x \cdot X_y = 1, G = X_y \cdot X_y = 2$$

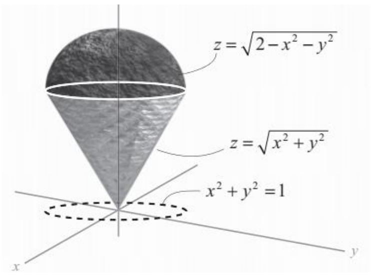
$$dA = \sqrt{EG - F^2} = \sqrt{3} \text{，} S = \iint_R dA = \int_0^2 \int_0^{2-y} \sqrt{3}\, dx dy = \ldots = 2\sqrt{3}$$

解 z=f(x,y)=2-x-y，$\frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = -1$

$$dS = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\, dA = \sqrt{3} dA$$

$$S = \iint_R \sqrt{3} dA = \sqrt{3} \int_0^2 \int_0^{2-x} dy dx = 2\sqrt{3}$$

Example 2



Find the surface area of the ice-cream cone $z = \sqrt{x^2 + y^2}$ that lies above the circular region $x^2 + y^2 \le 1$.

$$\varphi(r, \theta) = (r\cos\theta, r\sin\theta, r)$$
$$\varphi_r = (\cos\theta, \sin\theta, 1), \varphi_\theta = (-r\sin\theta, r\cos\theta, 0)$$
$$E = 2, F = 0, G = r^2$$
$$S = \iint_R dA = \int_0^{2\pi} \int_0^1 \sqrt{2r^2}\, dr d\theta = \ldots = \sqrt{2}\pi$$

另解

$$z = f(x,y) = \sqrt{x^2+y^2} \quad , \quad \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$dS = \sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}\, dA = \ldots = \sqrt{2}dA$$

$$S = \iint_R \sqrt{2}dA = \sqrt{2}\pi$$

Exercise

5. Set up the double integral for the area of the portion of the paraboloid $z = 16 - x^2 - y^2$ in the first octant.

6. Set up the double integral for the area of the surface $f(x, y) = 2y + x^2$ over the triangular region R with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

7. Set up the double integral in polar coordinates for the area of the surface $f(x, y) = 9 - x^2 - y^2$ over the region R given by $R = \{(x, y) : 0 \le f(x, y)\}$.

8. Set up the double integral for the area of the surface $f(x, y) = e^x$ over the region R given by $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$.

9. Show that the surface area of the cone $z = k\sqrt{x^2+y^2}$, $k > 0$, over the circular region $x^2 + y^2 \le r^2$ in the xy-plane is $\pi r^2\sqrt{k^2+1}$.