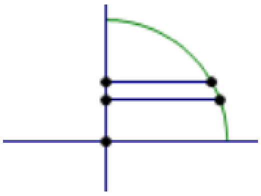
§ Lesson 21 Double Integral in Polar Coordinates

Example 1

$$\int_0^2\int_0^{\sqrt{4-y^2}} y\,dx\,dy =$$

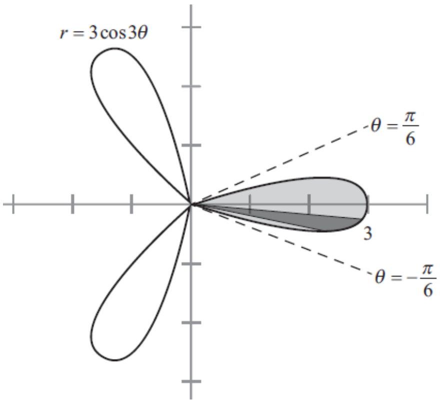


在半徑=2 的 1/4 圓上積分

Let $x = r\cos\theta, y = r\sin\theta$ ，then

$$\int_0^2\int_0^{\sqrt{4-y^2}} y\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^2 (r\sin\theta) r\,dr\,d\theta = \frac{8}{3}$$

Example 2



$r = 3\cos 3\theta$

Rose curve with 3 petals 計算其中一葉的面積

解 1

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{3\cos 3\theta} r\,dr\,d\theta = \frac{3\pi}{4}$$

解 2

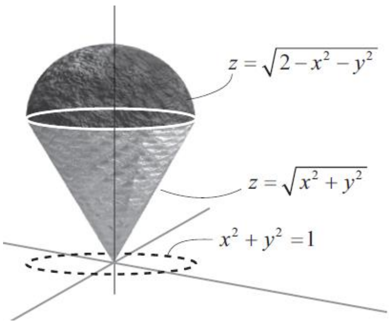
$$dA = \frac{1}{2}r^2 d\theta$$

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(3\cos 3\theta)^2 d\theta = \frac{9}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta\, d\theta$$

$$= \frac{9}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta = \frac{3\pi}{4}$$

Example 3

Set up the double integral for the volume of the ice-cream cone bounded above by the hemisphere $z = \sqrt{2-x^2-y^2}$ and bounded below by the cone $z = \sqrt{x^2+y^2}$.



解 1

The volume of the ice-cream

$$\sqrt{2-x^2-y^2} = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 1$$

Let $x = r\cos\theta, y = r\sin\theta$

$$V = \int_0^{2\pi}\int_0^1 (\sqrt{2-r^2} - r) r\,dr\,d\theta = \frac{4(\sqrt{2}-1)}{3}\pi$$

解 2

球面座標(spherical coordinates)

$x = r\sin\varphi\cos\theta, y = r\sin\varphi\sin\theta, z = r\cos\varphi$

$$dV = r^2 \sin\varphi dr d\varphi d\theta$$

$$0 \le r \le \sqrt{2}, 0 \le \varphi \le \frac{\pi}{4}, 0 \le \theta \le 2\pi$$

$$V = \iiint_{\Omega} dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} r^2 \sin\varphi dr d\varphi d\theta = \frac{4\pi(\sqrt{2}-1)}{3}$$

Exercise

6. Use a double integral in polar coordinates to find the area bounded by the three-leaved rose curve $r = 2\sin 3\theta$.

7. Use a double integral in polar coordinates to find the volume of the solid in the first octant bounded by $z = xy$ and $x^2 + y^2 = 1$.

8. Use a double integral in polar coordinates to find the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$, $z = 0$, and $x^2 + y^2 = 25$.

9. Use a double integral in polar coordinates to find the volume of the solid inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$.

10. Set up the double integral in polar coordinates for the area inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$.