Lesson 17 Lagrange Multipliers---Constrained Optimization

- Lagrange's theorem: Let f and g have continuous first partial derivatives such that f has an extremum at (x_0, y_0) on the smooth constraint curve $g(x, y) = k$. If $\nabla g(x_0, y_0) \neq 0$, then there is a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$. The number λ is called a **Lagrange multiplier**.

Example

Find the maximum value of the function $f(x, y) = 4xy$, where $x, y > 0$, subject to the constraint

1. $$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

$\nabla f = (4y, 4x), \nabla g = (\frac{2x}{9}, \frac{y}{8})$ ，there is a real number λ such

$\nabla f(x_0, y_0) = \lambda g(x_0, y_0)$ 解得 $x_0 = \frac{3\sqrt{2}}{2}, y_0 = 2\sqrt{2}$

And the maximum value is $f(\frac{3\sqrt{2}}{2}, 2\sqrt{2}) = 24$

2. Given that $x^2 + y^2 = 9$ ， $g(x, y) = x^3 - 3xy^2$ find the max and min

   Let $f(x, y, \lambda) = x^3 - 3xy^2 + \lambda(x^2 + y^2 - 9)$

   $$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 + 2\lambda x = 0$$

   $$\frac{\partial f}{\partial y} = -6xy + 2\lambda y = 0$$

   $$9x^2 - 3y^2 = 0$$

   Maximum=27 ，minimum=-27

Exercises

7. Use Lagrange multipliers to find the minimum distance from the parabola $y = x^2$ to the point $(0, 3)$.

8. A cargo container in the shape of a rectangular solid must have a volume of 480 cubic feet. The bottom will cost \$5 per square foot to construct, and the sides and the top will cost \$3 per square foot to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has minimum cost.