Lesson 15 Directional Derivatives and Gradients (梯度)

- Let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector in the plane, and let $f$ be a differentiable function of $x$ and $y$. The **directional derivative** of $f$ in the direction of $\mathbf{u}$ is

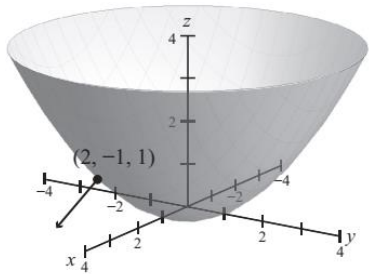
$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

- Let $z = f(x, y)$ be a function whose partial derivatives exist. The **gradient** of $f$ is the vector $\mathbf{grad}\, f(x, y) = \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$.

- Theorem: $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$.

u 是一數量場，u=constant 是一等值面(level surface) 則 grad u 垂直於通過該點的等值面，且指向函數 u(P)增大的方向。

Example




$f(x, y, z) = x^2 + y^2 - 4z$ ， $\nabla f = (2x, 2y, -4)$ ，

at the point(2,-1,1)，f(2,-1,1)=1，

the level surface(等值面) through the point is

$x^2 + y^2 - 4z = 1 \Rightarrow z = \frac{1}{4}(x^2 + y^2 - 1)$

the gradient is pointing downward at the point(2,-1,1) on the paraboloid。

例

在原點的點電荷為 q，則在 P(x,y,z)的電位 $v = \frac{q}{4\pi\epsilon r}, \vec{r} = (x, y, z), r = |\vec{r}|$

$grad r = \frac{\vec{r}}{r}$

電場強度 $E = \frac{q}{4\pi\epsilon r^3}\vec{r} = -grad v$

Exercises

**6.** Find the gradient of the function $f(x, y, z) = 3x^2 - 5y^2 + 2z^2$ at the point $(1, 1, -2)$.

**7.** Find the maximum value of the directional derivative of the function $f(x, y) = x^2 + 2xy$ at the point $(1, 0)$.

**8.** Find the maximum value of the directional derivative of the function $f(x, y, z) = xy^2z^2$ at the point $(2, 1, 1)$.

**9.** Find a normal vector to the level curve $f(x, y) = 6 - 2x - 3y = 6$ at the point $P(0, 0)$.

**10.** The temperature at the point $(x, y)$ on a metal plate is $T = \frac{x}{x^2 + y^2}$. Find the direction of greatest increase in heat from the point $(3, 4)$.

8. $grad f = \left(y^2 z^2, 2xyz^2, axy^2 z\right), grad f_P = (1,4,4), |grad f_P| = \sqrt{33}$