清大 108 博士班考題 高等微積分

1. The supremum property (completeness) of R means that every non-empty set of real numbers which has an upper bound has a supremum。

   Use this to show that there exists a positive number x such that $x^2 = 2$

   Consider the set $S = \{y \in R : y \geq 0, y^2 \leq 2\}$

(1) $1 \in S$ 所以 $S \neq \phi$

(2) S is bounded above 設 $\alpha = \sup S$ 。We show that $\alpha^2 = 2$ by contradiction

2.

2. (15 pts) Recall that the Cantor set F is the intersection of the sets $F_n, n \in \mathbb{N}$, obtained by succesive removal of open middle thirds, i.e., $F_1 = [0, 1/3] \cup [2/3, 1]$, $F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ and so on. (a) Show that every point in F has a ternary (base 3) expansion using only the digits $0, 2$. (b) Find the ternary point of $1/4$ and determine if $1/4$ belongs to the cantor set.

3. Let $A = \{x : x \in (0,1], \sin\frac{1}{x} = 0\} \cup \{0\}$ Is A a compact set？Prove it。

A=$\{\frac{1}{n\pi} | n = 1, 2, 3, \ldots\}$ Closed and bounded，so it is compact。

4. If $X = \{x_n\}$ be a bounded sequance in R and $\{\sigma_n\}$ is the sequence of arithmetic means。Prove that $\limsup\{\sigma_n\} \leq \limsup\{x_n\}$

   $$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

5. Does the integral $\int_0^\infty \frac{\sin x}{x} dx$ converge？ $y = f(x) = \frac{\sin x}{x}$ 稱為 sinc 函數

解 1. Laplace transform $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{\sin t}{t} (\lim_{s\to 0} e^{-st}) dt = \lim_{s\to 0} \int_0^\infty e^{-st} \frac{\sin t}{t} dt = \lim_{s\to 0} L\{\frac{\sin t}{t}\}$$

$$= \lim_{s\to 0} \int_s^\infty F(u) du \, , \, F(u) = L\{\sin t\} = \frac{1}{u^2+1}$$

$$= \lim_{s\to 0} \tan^{-1} u \Big|_s^\infty = \frac{\pi}{2}$$

解 2. 考慮 I(a)=$\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$

$$\frac{dI}{da} = \ldots = \int_0^\infty -e^{-ax} \sin x dx = -\frac{1}{a^2+1}$$

積分 $I(a) = -\arctan a + \frac{\pi}{2}$

Let $a \to 0^+$ 得 $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

6. Suppose that f is a continuous real valued function。

   Show that $\int_0^1 f(x)x^2 dx = \frac{1}{3} f(\xi)$ for some $\xi \in [0,1]$

均值定理 $\int_0^1 f(x)g(x)dx = f(\xi)\int_0^1 g(x)dx$ for some $\xi \in [0,1]$

7. Let $f(x,y) = e^x \cos y$。

   Find the Taylor expansion of f(x,y) around $(0,\frac{\pi}{2})$ to order 3 (Remainder term is the derivative of order 3) [GA3.1-2Laplacian]

   $$f(x,y) \approx \frac{\pi}{2} - y - \frac{1}{2}x(2y-\pi) + \frac{1}{4}x^2(\pi - 2y) + \frac{1}{48}(2y-\pi)^3$$

   $$f(x,y) = -(y-\frac{\pi}{2})(1 + x + \frac{x^2}{2}) + \frac{1}{6}(y-\frac{\pi}{2})^3$$

8. Let $f: R^3 \to R^2$ be defined by f(x,y,z)=(x+y+z,x-y-2xz)

   (a) Note f(0,0,0)=(0,0)。

   Show that we can solve for $(x,y) = \phi(z) = (\phi_1(z), \phi_2(z))$ near z=0

   (b) Find $D\phi(0) =$

   證明可以在 z=0 附近解出(x,y)為 z 的函數，即存在$\phi_1(z),\phi_2(z)$使得 $f(\phi_1(z),\phi_2(z),z) = (0,0)$

   Implicit function theorem

   1. F is continuously differentiable
   2. Jacobian matrix of w.r.t. (x,y) is invertible at (0,0)

   $$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-2z & -1 \end{pmatrix}$$

   At (x,y,z)=(0,0,0) the determinant is $\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$

   By Implicit function theorem there is a n,b,d, and a continuous differentiable function $\phi: R \to R^2$ such that $\phi(z) = (x(z), y(z))$ and f(x(z),y(z),z)=(0,0) for z near 0 with $\phi(0)=(0,0)$

   $$D\phi(0) = (\begin{matrix} \phi_1^{'}(0) \\ \phi_2^{'}(0) \end{matrix}) =$$

For $\frac{\partial f_1}{\partial z}$ $\frac{\partial f_1}{\partial x}\frac{dx}{dz}+\frac{\partial f_1}{\partial y}\frac{dy}{dz}+\frac{\partial f_1}{\partial d}=0$

For $\frac{\partial f_2}{\partial z}$ $\frac{\partial f_2}{\partial x}\frac{dx}{dz}+\frac{\partial f_2}{\partial y}\frac{dy}{dz}+\frac{\partial f_2}{\partial z}=0$

At (0,0,0) $\begin{cases}\frac{dx}{dz}+\frac{dy}{dz}=-1\\ \frac{dx}{dz}-\frac{dy}{dz}=0\end{cases}\Rightarrow D\phi(0)=\begin{pmatrix}-\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$