§ 夾擠(squeeze)原理

(一)

若$b_n \le a_n \le c_n$且$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \alpha$ 則$\lim\limits_{n\to\infty} a_n = \alpha$

例

$$a_n = \frac{1}{\sqrt{4n^2+1}} + \frac{1}{\sqrt{4n^2+2}} + \frac{1}{\sqrt{4n^2+3}} + \dots + \frac{1}{\sqrt{4n^2+n}}$$ ，求$\lim\limits_{n\to\infty} a_n = ?$

令$b_n = \frac{1}{\sqrt{4n^2+n}} + \frac{1}{\sqrt{4n^2+n}} + \dots + \frac{1}{\sqrt{4n^2+n}}$，$c_n = \frac{1}{\sqrt{4n^2+1}} + \frac{1}{\sqrt{4n^2+1}} + \dots + \frac{1}{\sqrt{4n^2+1}}$

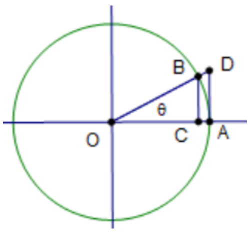
則$b_n \le a_n \le c_n$ 且$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \frac{1}{2}$

所以$\lim\limits_{n\to\infty} a_n = \frac{1}{2}$

(二)

若一區間，$g(x) \le f(x) \le h(x)$ 且$\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = \alpha$ 則$\lim\limits_{x\to a} f(x) = \alpha$

例 證明$\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



單位圓內，因為$\overline{BC} < AB < \overline{AD}$

(或者說$\Delta OBC <$扇形 OAB< ΔOAD)

$\sin\theta < \theta < \tan\theta$

則$\cos\theta < \frac{\sin\theta}{\theta} < 1$

因為$\lim\limits_{\theta\to 0}\cos\theta = 1$ 所以$\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

習作

1. 證明$\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$