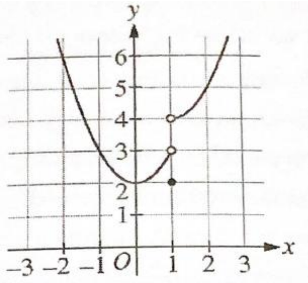
§ Limitation & extreme values



Graph of $f$

1. The graph of the function f is shown in the figure，The value $\lim_{x\to 0} f(1-x^2)=$

$$\lim_{x\to 0} f(1-x^2)=\lim_{x\to 1^-} f(x)=3$$

2. The derivative of the function of f is given by $f'(x)=e^{-x}\cos(x^2)$ ，for all real numbers x。What is the minimum value of f(x) for $-1\le x\le 1$

(A) f (-1) (B) f (-0.762) (C)f (1)

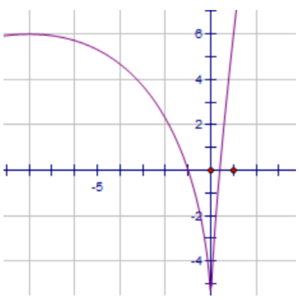
(D)There is no minimum value of f(x) for $-1\le x\le 1$

$$f(x)=\int_{-1}^{x} f'(t)dt=\int_{-1}^{x} e^{-t}\cos(t^2)dt$$

因為 $F(x)=e^{-x}\cos(x^2)>0$ and decreasing for $-1\le x\le 1$

所以 f(x)在 x=-1 時有最小值

3. The function f given by $f(x)=9x^{\frac{2}{3}}+3x-6$ has a relative minimum at x=

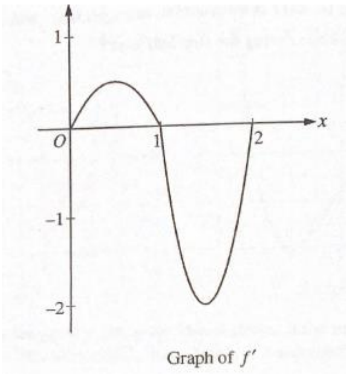



(A)-8 (B)$-\sqrt[3]{2}$ (C)-1 (D)$-\frac{1}{8}$ (E) 0

$f'(x)=6x^{-\frac{1}{3}}+3=0$ ，x=-8

$\lim_{x\to 0^-} f'(x)\to -\infty, \lim_{x\to 0^+} f'(x)\to \infty$

在 x=0 有 relative minimum




4. The figure left shows the graph of $f'$，the derivative f for $0 \le x \le 2$。What is the value of x at which the absolute minimum of f occurs？

(A)0 (B)$\frac{1}{2}$ (C)1 (D)$\frac{3}{2}$ (E) 2 E

$f'(x) = 0, x = 0,1,2$

$f(x) = \int_0^x f'(t)dt$，考慮 x=0,1,2

顯然 f(2)有最小值(∵圖形在 x 軸下方時 積分值<0)

§ Series

1. Let f be the function given by $f(x) = \frac{1}{2+x}$。What is the coefficient of $x^3$ in the Taylor series for f about x=0 ？

(A)$-\frac{3}{8}$ (B)$-\frac{1}{8}$ (C)$-\frac{1}{16}$ (D)$\frac{1}{24}$ (E)$\frac{1}{16}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$\frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{1}{2}x}\right) = \frac{1}{2}\{1 - \frac{1}{2}x + (\frac{1}{2}x)^2 - (\frac{1}{2}x)^3 + \ldots\}$$

So the coefficient f $x^3$ is $\frac{1}{2} \times (-\frac{1}{8}) = -\frac{1}{16}$

2. The function f has derivative of all orders for all real numbers，and $f^{(4)}(x) = e^{\sin x}$。If the third-degree Taylor polynomial for f about x=0 is used to approximate f on the interval [0,1]，what is the Lagrange error bound for the maximum error on the interval [0,1]？

(A)0.019 (B)0.097 (C)0.113 (D)0.399 (E) 0.417 B

$|R_n| \le \frac{f^{(n+1)}(z)|x-a|^{n+1}}{(n+1)!}$， z between a and x，a=0

$$\frac{e^{\sin 1}}{4!} \approx 0.0966$$

sin(1)==0.01745 sin(57.3)=0.8415

3. If the power series $\sum_{n=0}^{\infty} a_n (x-4)^n$ converges at x=7 and diverges at x=9，which of the following must be true？

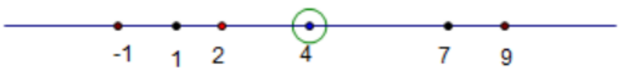
I. The series converges at x=1

II. The series converges at x=2

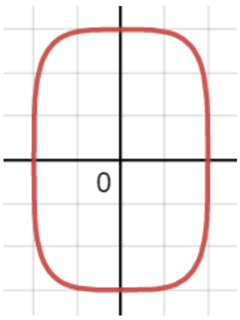
III. The series diverges at x=-1

(A)I only (B)II only (C)I and II only (D)II and III only B

$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-4)^{n+1}}{a_n(x-4)^n}\right|<1$ ，$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-4)^{n+1}}{a_n(x-4)^n}\right|=1$是不定狀況




§ integration




1. The base of a solid is the region enclosed by the curve $\frac{x^4}{16}+\frac{y^4}{81}=1$。For the solid，each cross section perpendicular to the x-axis is a semicircle。What is the volume of the solid？

(A)12.356 (B)15.732 (C)22.249 (C)24.712 (E)49.425

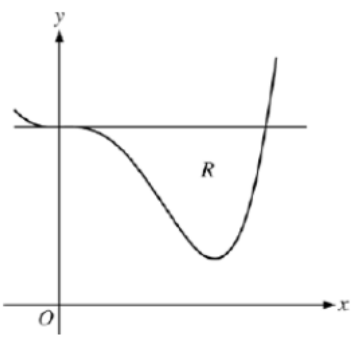
$$\int_{-2}^{2}\frac{1}{2}\pi y^2 dx = 9\pi\int_0^2\sqrt{1-\frac{x^4}{16}}dx = 9\pi\times 1.7481 \approx 49.426$$

2. Let $R$ be the region enclosed by the graph of $f(x)=x^4-2.3x^3+4$ and the horizontal line $y=4$, as shown in the figure above.

(a) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y=-2$.

(b) Region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is an isosceles right triangle with a leg in $R$. Find the volume of the solid.

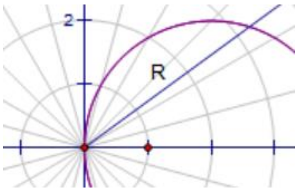
(c) The vertical line $x=k$ divides $R$ into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value $k$.




(b) $\int_0^{2.3}\frac{1}{2}(4-f(x))^2 dx = 3.574$ 2014 年

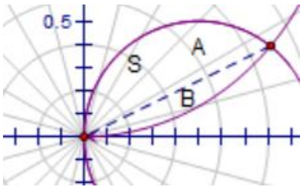
§ Polar coordinate

1. Let R be the region in the first quadrant that is bounded above by the polar curve $r=4\cos\theta$ and below the line $\theta=1$，what is the area of R？




$$\frac{1}{2}\int_{1}^{\frac{\pi}{2}} r^2 d\theta=\frac{1}{2}\int_{1}^{\frac{\pi}{2}}(4\cos\theta)^2 d\theta=0.4645...$$

2. Let R be the region in the first quadrant that is bounded above by the polar curve $r=\cos\theta$ and bounded below by the graph of the polar curve $r=2\theta$，the two curves intersect when $\theta=0.450$。What is the area of S？




A+B=$\frac{1}{2}\int_{0.450}^{\frac{\pi}{2}}\cos^2\theta d\theta+\frac{1}{2}\int_{0}^{0.450}(2\theta)^2 d\theta$

=0.18228+0.0607=0.243

§ parameter equation & application in physics

1. At time $t\geq 0$，a particle moving in the xy-plane has the velocity vector given by $v(t)=<3,2^{-t^2}>$。If the particle at the point $(1,\frac{1}{2})$ at time t=0，how far is the particle from the origin at time t=1？

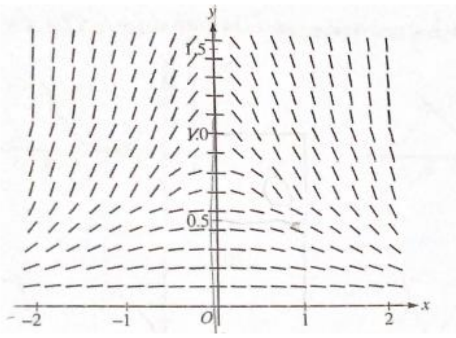
(A)2.304 (B)3.107 (C)4.209 (D)5.310

x(t)=3t+1，x(1)=4

$$y(1)=\int_{0}^{1}2^{-t^2}dt+\frac{1}{2}=0.810+0.5=1.310$$

$$\sqrt{16+(1.310)^2}=4.209$$

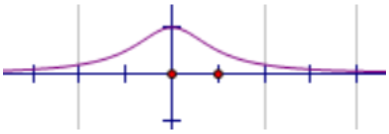
§ slope field




The slope field for a certain differential equation is shown as left。Which of the following could be a solution to the differential equation with initial condition y(0)=1

(A)y=cos x (B) $y=1-x^2$ (C) $y=e^x$

(D) $y=\sqrt{1-x^2}$ (E) $y=\dfrac{1}{1+x^2}$ (E)



$y=\dfrac{1}{1+x^2}$

§ Differential equation

1. The number of student in a cafeteria is modeled by the function P that satisfies the logistic differential equation $\dfrac{dP}{dt}=\dfrac{1}{2000}P(200-P)$ ，where t is the time in seconds and P(0)=25。What is the greatest rate of change，in students per seconds，of the number of students in the cafeteria？

   (A)5 (B)25 (C)100 (D)200

   $P(200-P)=-P^2+200P=-(P-100)^2+10000$

   $\dfrac{dP}{dt}$的最大值=$\dfrac{1}{2000}\times 10000=5$

2. The number of students in a school who have heard a rumor at time t hours is modeled by function P，the solution to a logistic differential equation。At noon，50 of the school's 500 students have heard the rumor，also at noon ，P is increasing at a rate of 20 students per hour。Which of the following could be the logistic differential equation？

   (A) $\dfrac{dP}{dt}=\dfrac{1}{1125}P(500-P)$ (B) $\dfrac{dP}{dt}=\dfrac{1}{480}P(500-P)$ (C) $\dfrac{dP}{dt}=\dfrac{1}{192}P(500-P)$

   (D) $\dfrac{dP}{dt}=\dfrac{2}{45}P(500-P)$ (E) $\dfrac{dP}{dt}=\dfrac{5}{48}P(500-P)$

設 $\frac{dP}{dt} = kP(500-P)$ ，P(0)=50 at noon， $\frac{dP}{dt} = 20$ at t=0

$20 = k \times 50 \times (500-50)$ ， $k = \frac{1}{1125}$

So the answer is (A)

1. If $f(x) = \int_1^{x^3} \frac{1}{1+\ln t}\, dt$ for $x \geq 1$, then $f'(2) =$

$\frac{12}{1+\ln 8}$

2. Which of the following limits is equal to $\int_3^5 x^4\, dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{2}{n}$

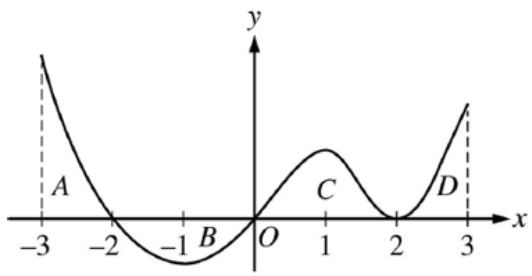
3. Let $y = f(t)$ be a solution to the differential equation $\frac{dy}{dt} = ky$, where $k$ is a constant. Values of $f$ for selected values of $t$ are given in the table above. Which of the following is an expression for $f(t)$?

(A) $4e^{\frac{t}{2}\ln 3}$

(B) $e^{\frac{t}{2}\ln 9} + 3$

(C) $2t^2 + 4$

(D) $4t + 4$




Graph of $f$

The graph of a differentiable function $f$ is shown above for $-3 \le x \le 3$. The graph of $f$ has horizontal tangent lines at $x = -1$, $x = 1$, and $x = 2$. The areas of regions $A$, $B$, $C$, and $D$ are 5, 4, 5, and 3, respectively. Let $g$ be the antiderivative of $f$ such that $g(3) = 7$.

(a) Find all values of $x$ on the open interval $-3 < x < 3$ for which the function $g$ has a relative maximum. Justify your answer.

(b) On what open intervals contained in $-3 < x < 3$ is the graph of $g$ concave up? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 0} \frac{g(x)+1}{2x}$, or state that it does not exist. Show the work that leads to your answer.

(d) Let $h$ be the function defined by $h(x) = 3f(2x+1)+4$. Find the value of $\int_{-2}^{1} h(x)\,dx$.

4.

(a) $g$ has a relative maximum at $x = -2$ since $g' = f$ changes sign from positive to negative at $x = -2$.

(b) The graph of $g$ is concave up for $-1 < x < 1$ and $2 < x < 3$ because $g' = f$ is increasing on those intervals.

(c) Because $g$ is continuous at $x = 0$, $\lim_{x \to 0} g(x) = g(0)$.

$$g(3) = g(0) + \int_0^3 f(x)\,dx$$

$$g(0) = g(3) - \int_0^3 f(x)\,dx = 7 - (5+3) = -1$$

$\lim_{x \to 0} g(x) + 1 = 0$ and $\lim_{x \to 0} 2x = 0$.

Using L'Hospital's Rule,

$$\lim_{x \to 0} \frac{g(x)+1}{2x} = \lim_{x \to 0} \frac{g'(x)}{2} = \lim_{x \to 0} \frac{f(x)}{2} = \frac{f(0)}{2} = 0$$

(d)25.5

The position of a particle moving in the $xy$-plane is given by the parametric equations $x(t) = \frac{6t}{t+1}$ and $y(t) = \frac{-8}{t^2+4}$. What is the slope of the line tangent to the path of the particle at the point where $t = 2$?

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{3}{4}$

(D) $\frac{4}{3}$

5.

C

For what value of $k$, if any, is $\int_0^\infty kxe^{-2x}\,dx = 1$?

(A) $\frac{1}{4}$

(B) 1

(C) 4

(D) There is no such value of $k$.

6.

C

The Taylor series for a function $f$ about $x = 0$ converges to $f$ for $-1 \le x \le 1$. The $n$th-degree Taylor polynomial for $f$ about $x = 0$ is given by $P_n(x) = \sum_{k=1}^{n} (-1)^k \frac{x^k}{k^2+k+1}$. Of the following, which is the smallest number $M$ for which the alternating series error bound guarantees that $|f(1) - P_4(1)| \le M$?

(A) $\frac{1}{5!} \cdot \frac{1}{31}$

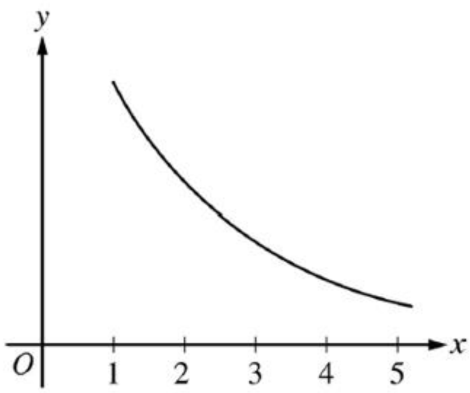
(B) $\frac{1}{4!} \cdot \frac{1}{21}$

(C) $\frac{1}{31}$

(D) $\frac{1}{21}$

7.

C




Graph of $g$

For $x \geq 1$, the continuous function $g$ is decreasing and positive. A portion of the graph of $g$ is shown above. For $n \geq 1$, the $n$th term of the series $\sum_{n=1}^{\infty} a_n$ is defined by $a_n = g(n)$. If $\int_1^{\infty} g(x)\,dx$ converges to 8, which of the following could be true?

8. (A) $\sum_{n=1}^{\infty} a_n = 6$ (B) $\sum_{n=1}^{\infty} a_n = 8$ (C) $\sum_{n=1}^{\infty} a_n = 10$ (D) $\sum_{n=1}^{\infty} a_n$ $diverges$

C

The function $f$ has derivatives of all orders at $x = 0$, and the Maclaurin series for $f$ is $\sum_{n=2}^{\infty} \frac{\ln n}{3^n n^3} x^n$.

(a) Find $f'(0)$ and $f^{(4)}(0)$.

(b) Does $f$ have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(c) Using the ratio test, determine the interval of convergence of the Maclaurin series for $f$. Justify your answer.

9.

(a) $\frac{f'(0)}{1!} = a_1 = 0 \;\Rightarrow\; f'(0) = 0$

$$\frac{f^{(4)}(0)}{4!} = a_4 = \frac{\ln 4}{3^4 4^3} \;\Rightarrow\; f^{(4)}(0) = \frac{\ln 4}{3^4 4^3} \cdot 4! = \frac{\ln 4}{216}$$

(b) $f'(0) = 0$

$$\frac{f''(0)}{2!} = a_2 = \frac{\ln 2}{3^2 2^3} \Rightarrow f''(0) = \frac{\ln 2}{3^2 2^3} \cdot 2! = \frac{\ln 2}{36} > 0$$

By the Second Derivative Test, $f$ has a relative minimum at $x = 0$.

(c) Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\dfrac{\ln(n+1)}{3^{n+1}(n+1)^3} x^{n+1}}{\dfrac{\ln n}{3^n n^3} x^n} \right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln n} \cdot \left(\frac{n}{n+1}\right)^3 \cdot \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1$$

$|x| < 3$, therefore the radius of convergence is $R = 3$, and the series converges on the interval $-3 < x < 3$.

When $x = 3$, the series is $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$.

Because $0 < \frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for all $n \geq 2$ and the $p$-series

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges by the

comparison test.

When $x = -3$, the series is $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n^3}$.

This series is absolutely convergent because $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges.

The interval of convergence is $-3 \leq x \leq 3$.

10. Which of the following series converge?

I.) $\sum_{n=1}^{\infty} \frac{n^2+3}{n^2}$, II.) $\sum_{n=1}^{\infty} \frac{2n!}{10^n}$, III.) $\sum_{n=1}^{\infty} \frac{3^n}{4^n+1}$

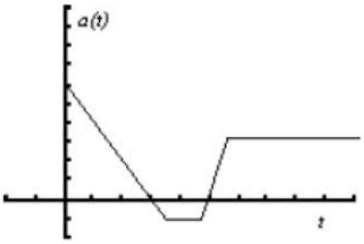
(A)I only (B)II only (C)III only (D)I, III only (E)II , III only

C

11. $\int_1^\infty \frac{dx}{4+9x^2} =$

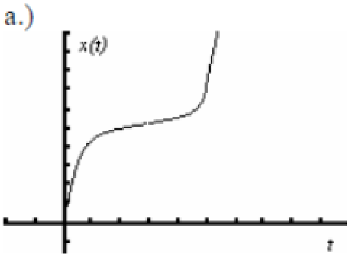
$$\frac{\pi}{12} - \frac{1}{6}\tan^{-1}\frac{3}{2}$$

.) The following graph represents the acceleration of an object:

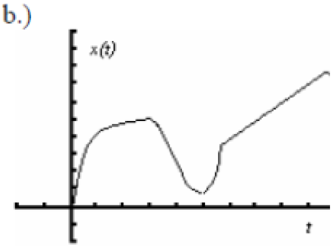



12. Which of the following graphs could represent the position of the object?

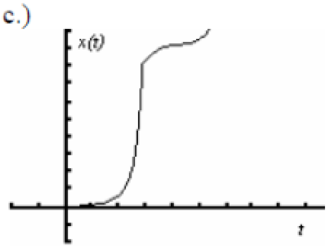
a.)



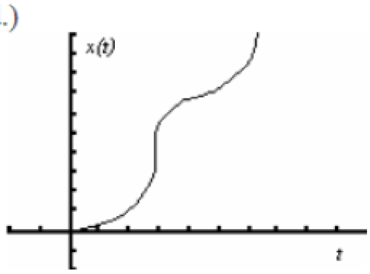

b.)



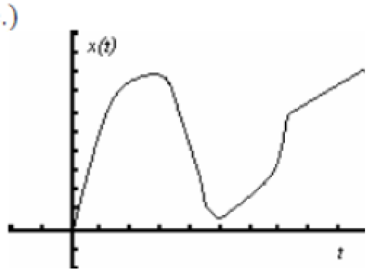

c.)




d.)




e.)




D

22.) The length of a curve on the interval [0, 5] is given as $\int_0^5 \sqrt{1+20x^8}\,dx$. If this particular curve contains the point (1, 3), which of the following could be an equation for this curve?

a.) $y = \frac{2\sqrt{5}}{5}x^5 + \frac{14}{5}$

b.) $y = \frac{20}{9}x^9 + \frac{7}{9}$

c.) $y = \frac{2\sqrt{5}}{5}x^5 + \frac{15-2\sqrt{5}}{5}$

d.) $y = \frac{2\sqrt{5}}{5}x^5 + \frac{15}{2\sqrt{5}}$

13. e.) $y = 2x^4 + 1$

B

14. Calculate the area between the curves $y=e^x$ and $y=e^{3x}$ ， bounded by the lines y=1 and y=4 $\frac{8}{3}\ln 4-2$

15. $\int x\cdot 3^{x^2}dx=$ $\frac{3^{x^2}}{2\ln 3}+C$

16. Given $f(x)=\int_1^{5x^3}\sin t dt$ ， $f'(x)=$ $15x^2\sin(5x^3)$

39.) Which of the following are $\theta$-values at which there are horizontal tangents of $r=4\theta\sin\theta$ on the interval $[0,2\pi]$? (Note: Not *all* of the values may be listed.)

17. I.) $\theta=0$, II.) $\theta=\pi$, III.) 5.6642

(A)I only (B)II only (C)III only (D)I ,II ,III (E)None of the above B

42.) The rate of change of a quantity is given by the differential equation $\frac{dy}{dt}=\frac{6t+3}{t^2+2t-3}$. If $y(2)=7$, which of the following is the value of the constant term in the equation for $y$?

18. a.) $7-\frac{15}{4}\ln 5$, b.) $7-\frac{9}{4}\ln 5$, c.) $7-\frac{15}{4}\ln 2$, d.) $7-\frac{9}{4}\ln 2$, e.) $7-\frac{15}{4}\ln 7$

A