

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \leq t \leq 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

- (a) How many people enter the line for the escalator during the time interval $0 \leq t \leq 300$?
- (b) During the time interval $0 \leq t \leq 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?
- (c) For $t > 300$, what is the first time t that there are no people in line for the escalator?
- (d) For $0 \leq t \leq 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

Escalator 自動扶梯

ANS

$$(a) \int_0^{300} r(t) dt = 270$$

$$(b) 20 + \int_0^{300} (r(t) - 0.7) dt = 20 + \int_0^{300} r(t) dt - 0.7 \cdot 300 = 80$$

$$(c) 300 + \frac{80}{0.7} \approx 414.286$$

- (d) The total number of people in line at time t , $0 \leq t \leq 300$, is modeled by $20 + \int_0^t r(x) dx - 0.7t$.

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

2. A particle moves along the x -axis with velocity given by $v(t) = \frac{10 \sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \leq t \leq 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

- (a) Find the acceleration of the particle at time $t = 3$.
- (b) Find the position of the particle at time $t = 3$.
- (c) Evaluate $\int_0^{3.5} v(t) dt$, and evaluate $\int_0^{3.5} |v(t)| dt$. Interpret the meaning of each integral in the context of the problem.
- (d) A second particle moves along the x -axis with position given by $x_2(t) = t^2 - t$ for $0 \leq t \leq 3.5$. At what time t are the two particles moving with the same velocity?

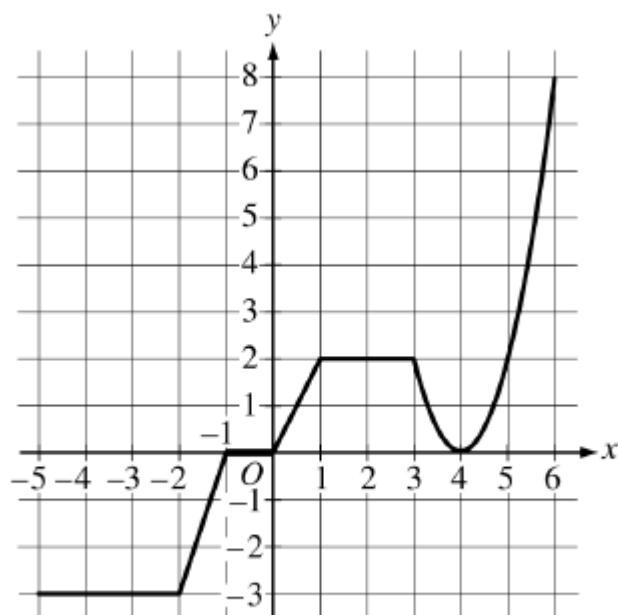
ANS

$$(a) v'(3) = -2.118$$

$$(b) x(3) = x(0) + \int_0^3 v(t) dt = -1.760213$$

$$(c) \text{位移} \int_0^{3.5} v(t) dt = 2.844 \quad \text{總距離} = \int_0^{3.5} |v(t)| dt = 3.737$$

$$(d) v(t) = x_2'(t) = 2t - 1, \quad t = 1.57$$

Graph of g

3. The graph of the continuous function g , the derivative of the function f , is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \leq x \leq 6$.
- If $f(1) = 3$, what is the value of $f(-5)$?
 - Evaluate $\int_1^6 g(x) dx$.
 - For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.
 - Find the x -coordinate of each point of inflection of the graph of f . Give a reason for your answer.

ANS

(a) 12.5

(b) 10

- The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.
- The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

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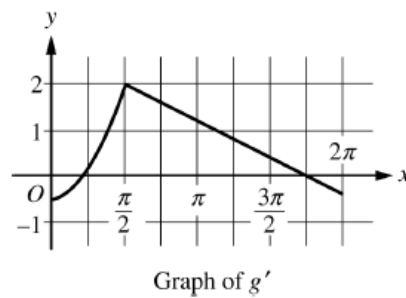
(a) $H'(6) \approx \frac{5}{2}$

(b) Mean Value Theorem

(c) $\frac{1}{10-2} \int_2^{10} H(t) dt \approx \frac{263}{32}$

(d) chain rule $\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$

5. Let f be the function defined by $f(x) = e^x \cos x$.
- (a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.
- (b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?
- (c) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.
- (d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g' , the derivative of g , is shown below. Find the value of $\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.

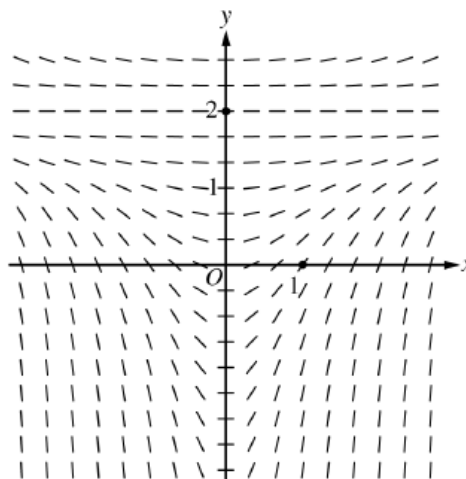


ANS

- (a) $\frac{-e^\pi - 1}{\pi}$
- (b) $e^{\frac{3\pi}{2}}$
- (c) $-\frac{1}{\sqrt{2}}e^{\frac{5\pi}{4}}$
- (d) $\frac{-e^{\frac{\pi}{2}}}{2}$

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

- (a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



- (b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

ANS

$$(a) \frac{-1}{y-2} = \frac{1}{6}x^2 + c$$

$$(b) y = \frac{4}{3}(x-1), \quad f(0.7) \approx -0.4$$

$$(c) y = 2 - \frac{6}{x^2 + 2}$$

BC

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \leq h \leq 30$ and is modeled by $f(h)$ for $h \geq 30$. The continuous function f is not explicitly given.
- (a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.
- (b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?
- (c) There is a function u such that $0 \leq f(h) \leq u(h)$ for all $h \geq 30$ and $\int_{30}^{\infty} u(h) dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.
- (d) The boat is moving on the surface of the sea. At time $t \geq 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662 \sin(5t)$ and $y'(t) = 880 \cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \leq t \leq 1$.

Plankton 浮游生物

ANS

(a) $p'(25) = -1.179$

(b) $dV = 3dh$, 質量 $dM = \text{體積} \times \text{密度} = 3p(h)dh$ $M = \int_0^{30} 3p(h)dh = 1675.41$

- (c) $\int_{30}^K 3f(h) dh$ represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of K meters.

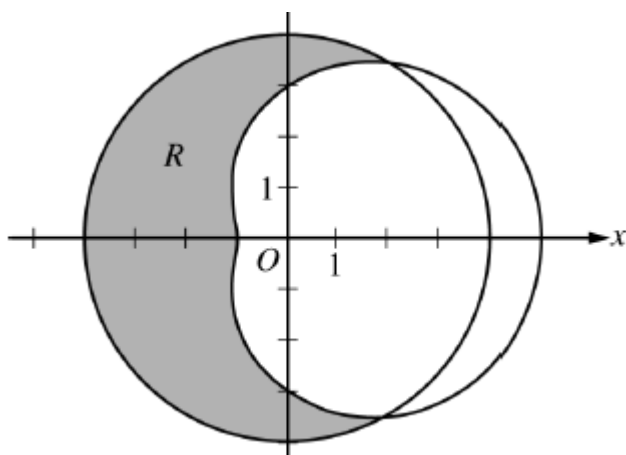
The number of plankton cells, in millions, in the entire column of water is given by $\int_0^{30} 3p(h) dh + \int_{30}^K 3f(h) dh$.

Because $0 \leq f(h) \leq u(h)$ for all $h \geq 30$,

$$3 \int_{30}^K f(h) dh \leq 3 \int_{30}^K u(h) dh \leq 3 \int_{30}^{\infty} u(h) dh = 3 \cdot 105 = 315.$$

The total number of plankton cells in the column of water is bounded by $1675.415 + 315 = 1990.415 \leq 2000$ million.

$$(d) \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} dt = 757.455862$$



5. The graphs of the polar curves $r = 4$ and $r = 3 + 2 \cos \theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.
- (a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2 \cos \theta$, as shown in the figure above. Write an expression involving an integral for the area of R .
- (b) Find the slope of the line tangent to the graph of $r = 3 + 2 \cos \theta$ at $\theta = \frac{\pi}{2}$.
- (c) A particle moves along the portion of the curve $r = 3 + 2 \cos \theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

ANS

$$(a) \text{ Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} (4^2 - (3 + 2 \cos \theta)^2) d\theta$$

$$(b) \left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{2}} = \frac{2}{3}$$

$$(c) \text{chain rule } \left. \frac{d\theta}{dt} \right|_{\theta = \frac{\pi}{3}} = -\sqrt{3}$$

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

- (a) Write the first four nonzero terms and the general term of the Maclaurin series for f .
- (b) Determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.
- (c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

(b)
$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$$\left| \frac{x}{3} \right| < 1 \text{ for } |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

- (c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$